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TODA AND KDV

DAVID GIESEKER

Abstract

The main object of this paper is to produce a deformation of the KdV hierarchy of partial differential equations. We construct this deformation by taking a certain limit of the Toda hierarchy. This construction also provides a deformation of the Virasoro algebra.

1. Introduction

Our aim in this paper is to produce a deformation of the KdV hierarchy whose existence was conjectured in [5]. To describe KdV algebraically following Gelfand and Dickey [2], let

$$R_0 = \mathbf{C}[w^{(0)}, w^{(1)}, \dots]$$

be a polynomial ring in infinitely many variables. Introduce a $\mathbf{C}$ derivation ∂ on R_0 by

$$\partial w^{(k)} = w^{(k+1)}.$$

An element of R_0 is intended to represent an abstract differential operator in one variable. If f is a $\mathcal{C}^\infty$ function on $\mathbf{R}$, then define

$$P(f) = P\left(f, \frac{df}{dx}, \dots\right),$$

i.e., substitute f for $w^{(0)}$, $\frac{df}{dx}$ for $w^{(1)}$, etc.

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To describe translationally invariant PDE's algebraically, consider $\mathbf{C}$ derivations D of R_0 which commute with ∂. The set of such D is naturally just R_0 under the correspondence $D \to D(w^{(0)})$. So R_0 inherits the structure of a Lie algebra, since the commutator of two derivations is a derivation. Let

$$K_1 = w^{(3)} + w^{(1)} w^{(0)},$$

called the KdV element of R_0. One of the main results of KdV theory is that K_1 lies in a large abelian subalgebra of R_0. In fact, there is a sequence of elements K_n of R_0 so that K_{n+1} is not in the linear span generated by the lower $K_n, \dots, K_1$ and all the K_k commute. These K_k are called the KdV hierarchy.

The main object of this paper will be to produce interesting deformations of the KdV hierarchy. That is, we seek to produce mutually commuting $L_k \in R_0[[\epsilon]]$ which become the K_k when we set $\epsilon = 0$, where $R_0[[\epsilon]]$ is the formal power series ring in ϵ.

While the ring R_0 captures much of the algebraic structure of KdV, sometimes one considers solutions or representations of KdV. Suppose that $\mathcal{M}$ is an analytic manifold and let $P_0 = \partial, P_1, \dots$ be derivations of R_0 commuting with ∂. Let $\chi_0, \chi_1, \dots,$ be vector fields on $\mathcal{M}$ and let f be a function on $\mathcal{M}$. We say that $f, \chi_0, \chi_1, \dots,$ form a representation of $P_0, P_1, \dots,$ if

$$\chi_k(f) = P_k(f, \chi(f), \chi^2(f), \dots),$$

where we regard the vector field χ_k as a derivation on functions on $\mathcal{M}$ and $\chi = \chi_0$.

The main example of representations of the K_k is the following: Let X be a hyperelliptic curve and let Q be a Weirstrass point. Let

$$\vartheta : H^1(\mathcal{O}_X) \to \mathbf{C}$$

be the theta function. Then we can find translationally invariant vector fields, $\chi_0, \chi_1, \dots,$ so that if

$$f = \chi_0^2(\log \vartheta),$$

then $f, \chi_0, \dots,$ form a representation of the KdV hierarchy.

Our aim is to develop a difference version of KdV hierarchy to obtain a deformation of the KdV hierarchy. Here the idea is basically to discretize a differential equation. The heuristic motivation for these discretizations in given in [5]. To be rigorous we need a method to describe difference equations. We consider the ring

$$S_1 = \mathbf{C}[\dots, X_{-1}, X_0, X_1, \dots; \dots, Y_{-1}, Y_0, Y_1, \dots]$$

and let

$$S_2 = \mathbf{C}[\dots, a_{-1}, a_0, a_1, \dots; \dots, b_{-1}, b_0, b_1, \dots].$$

Let $T : S_2 \to S_2$ be the $\mathbf{C}$ algebra homomorphism defined by $T(a_n) = a_{n+1}$ and $T(b_n) = b_{n+1}$. Now given $P, Q \in S_1$, we can define a derivation $D_{P,Q} : S_2 \to S_2$ by

$$D_{P,Q}(a_n) = P(\dots, a_{n-1}, \hat{a}_n, a_{n+1}, \dots; \dots, b_{n-1}, \hat{b}_n, b_{n+1}, \dots)$$

and

$$D_{P,Q}(b_n) = Q(\dots, a_{n-1}, \hat{a}_n, a_{n+1}, \dots; \dots, b_{n-1}, \hat{b}_n, b_{n+1}, \dots),$$

where the $\hat{}$ indicates that a_n should be substituted for X_0 and b_n should be substituted for Y_0. This construction gives all the derivations of S_2 commuting with T and so introduces a Lie algebra structure on $S_1 \oplus S_1$. The interesting example is the Toda equations:

$$T_1 = (P_1, Q_1) = (Y_{-1} - Y_0, Y_0(X_0 - X_1)).$$

The main theorem here due to Toda, Flaschka and many others is that T_1 lies in an unexpectedly large Abelian sub-algebra of the Lie algebra $S_1 \oplus S_1$. In fact, there is a whole sequence of mutually commuting $T_k \in S_1 \oplus S_1$.

We can also describe solutions of the Toda hierarchy using algebraic geometry following van Moerbeke. Let N be a positive integer. Let $\mathcal{C}$ be the space of all complex valued functions on $\mathbf{Z}$. Let $T : \mathcal{C} \to \mathcal{C}$ be translation by N, $T(f)(n) = f(n+N)$. Let $\mathcal{C}_N$ be the set of translation invariant functions: $T(f) = f$. Given A and B in $\mathcal{C}_N$, we define

$$L_{(A,B)} : \mathcal{C} \to \mathcal{C}$$

by the formula

$$L_{(A,B)}(\psi)(n) = \psi(n+1) + A(n)\psi(n) + B(n)\psi(n-1).$$

Thus $L_{(A,B)}(\psi)$ is a second order linear difference operator. By definition of $\mathcal{C}_N$, the operators $L_{(A,B)}$ and T commute, so we can reasonably look for common eigenfunctions of these two operators. If you think of $L_{(A,B)}$ as a discrete analogue of a Schrödinger operator, this amounts to finding the energy levels with a given quasi-momentum of a particle traveling through a periodic potential, a problem frequently encountered in solid state physics [1]. We then define the Bloch spectrum $\mathcal{B}_{(A,B)}$ of

$L_{(A,B)}$ to be the set of $(\lambda, \alpha) \in \mathbf{C} \times \mathbf{C}^*$ so that there is a nonzero function ψ with $L_{(A,B)}(\psi) = \lambda\psi$ and $T(\psi) = \alpha\psi$. By projecting to the λ axis, it is easy to see that $\mathcal{B}_{(A,B)}$ is a hyperelliptic curve, possibly singular. Indeed, there are in general two values of α associated to any fixed λ. $\mathcal{B}_{(A,B)}$ can be compactified to a curve $\overline{\mathcal{B}}_{(A,B)}$ by adding two points P and Q over $\lambda = \infty$. It turns out that the divisor $N(P - Q)$ is linearly equivalent to zero. It also turns out that $\mathcal{B}_{(A,B)}$ does not determine (A, B). There are interesting ways of moving (A, B) so that $\mathcal{B}_{(A,B)}$ remains fixed. Such a deformation of (A, B) keeping the Bloch spectrum fixed is called an isospectral deformation. The set of all (A', B') isospectral to (A, B) turns out to be isomorphic to the Jacobian of $\overline{\mathcal{B}}_{(A,B)}$ in a birational sense for generic A and B. In particular, any linear flow on the Jacobian becomes a non-linear flow on $\mathcal{C}_N \times \mathcal{C}_N$, which in turn is a linear combination of Toda flows. For example, there is a linear flow on the Jacobian so that if (A_t, B_t) indicates the flow of (A_0, B_0) after time t, then

$$\frac{dA_t(k)}{dt} = B_t(k-1) - B_t(k)$$

and

$$\frac{dB_t(k)}{dt} = B_t(k)(A_t(k) - A_t(k+1)).$$

Conversely, given a hyperelliptic curve C of genus g and two points P and Q on C and a suitably generic line bundle $\mathcal{L}$ of degree g so that $N(P - Q)$ is linearly equivalent to zero, we can define $A_{\mathcal{L}}$ and $B_{\mathcal{L}}$ in $\mathcal{C}_N$ so that linear flows on the Jacobian become Toda flows on $\mathcal{C}_N \times \mathcal{C}_N$. (See [7] for instance.)

[5] developed a method for taking the limit of Toda equations. Our purpose in this paper is to develop an algebraic framework for these limits and to prove the main conjecture of [5]. Here is a formalism to allow us to make sense of taking a limit of Toda equations. Let

$$R = \mathbf{C}[v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots].$$

Again we introduce a derivation ∂ by $\partial v^{(k)} = v^{(k+1)}$ and by considering R_0 as a subring of R. We think of elements $P \in R$ as being differential expressions in two functions f and g:

$$P(f, g) = P\left(f, \frac{df}{dx}, \ldots; g, \frac{dg}{dx}, \ldots\right).$$

For $k \in \mathbf{Z}$, let $E_k : R[[\epsilon]] \to R[[\epsilon]]$ be defined by

$$E_k = \exp(k\epsilon\partial)$$

as a formal power series in ϵ. Now suppose we have $(P_1, P_2) \in S_1 \oplus S_1$. We can then define a derivation $\mathcal{D}'_{P_1,P_2} : R[[\epsilon]] \to R[[\epsilon]]$ commuting with ∂ and continuous in ϵ topology by

$$(1)\qquad \mathcal{D}'_{P_1,P_2}(v^{(0)}) = P_1(\ldots, E_{-1}(v^{(0)}), \hat{E}_0(v^{(0)}), E_1(v^{(0)}), \ldots; \ldots, E_{-1}(w^{(0)}), \hat{E}_0(w^{(0)}), E_1(w^{(0)}), \ldots)$$

and

$$(2)\qquad \mathcal{D}'_{P_1,P_2}(w^{(0)}) = P_2(\ldots, E_{-1}(v^{(0)}), \hat{E}_0(v^{(0)}), E_1(v^{(0)}), \ldots; \ldots, E_{-1}(w^{(0)}), \hat{E}_0(w^{(0)}), E_1(w^{(0)}), \ldots),$$

where the $\hat{}$ is again a place holder. Define a **C** algebra endomorphism of $R[[\epsilon]]$ commuting with ∂ by

$$\Phi(v^{(0)}) = -2 + \epsilon^2 v^{(0)}$$

and

$$\Phi(w^{(0)}) = 1 + \epsilon^2 w^{(0)}.$$

Φ does not have an inverse on $R[[\epsilon]]$, but does have one on $R((\epsilon))$, the ring of Laurent series in ϵ which contain only finitely many negative powers of ϵ and finally define

$$\mathcal{D}_{P_1,P_2} = \Phi \mathcal{D}'_{P_1,P_2} \Phi^{-1}.$$

In particular, it turns out that we get a series of mutually commuting derivations

$$(3)\qquad \mathcal{D}_{T_k}$$

of $R[[\epsilon]]$ coming from the Toda hierarchy.

Suppose that $P \in R_0[[\epsilon]]$ and consider the element $W = v^{(0)} - P$ and let $\mathcal{I}_P$ be the closure of the ideal of $R[[\epsilon]]$ generated by W, ∂W, $\partial^2 W$, etc. Notice that $R[[\epsilon]]/\mathcal{I}_P$ is naturally isomorphic to $R_0[[\epsilon]]$. The following is one of our main results:

Theorem 1.0.1. *There is a P so that $\mathcal{I}_P$ is invariant under all the $\mathcal{D}_{T_k}$. Thus the $\mathcal{D}_{T_k}$ induce derivations $\mathbf{D}_{T_k}$ of $R_0[[\epsilon]]$. Suitable linear combinations of the $\mathbf{D}_{T_k}$ over $\mathbf{C}((\epsilon))$ are a deformation of the* KdV *hierarchy.*

We construct this P recursively in powers of ϵ. Suppose we have found a P which works to order ϵ^n and we write $P' = P + \epsilon^n P_1$. For $\mathcal{D}_{T_1}$ to preserve $\mathcal{I}_{P'}$ to order ϵ^{n+1} it turns out that there is an element Q_n computed in terms of the original P so that $\partial P_1 = Q_n$. But it is not at all obvious why Q_n should be the total derivative of anything. Thus at each stage of constructing P, we meet a highly nontrivial obstruction. We will show that there are lots of functions g so that

$$\int_z^{z+1} Q_n(g) = 0 \tag{4}$$

for generic $z \in \mathbf{C}$. If (4) is true for generic enough g, then Q_n is a derivative.

We define a generalized idea of representations. Suppose that we have an element $P \in R$. Let $\mathcal{M}$ be an analytic manifold and let χ be an analytic vector field on $\mathcal{M}$ and let f and g be two meromorphic functions on $\mathcal{M}$. We define $P_\chi(f, g)$ to be $P(f, \chi f, \chi^2 f, \dots, g, \chi g, \chi^2 g, \dots)$. Next, suppose we have a function h on $\mathcal{M}$. For any given $N \in \mathbf{Z}^+$, we can extend this definition to $P \in R[[\epsilon]]$ by

$$P_{\chi,N}(\sum_{n=0}^{\infty} P_n \epsilon^n)(f, g) = \sum_{n=0}^{N} h^n P_{n,\chi}(f, g).$$

If h_1 and h_2 are two functions on $\mathcal{M}$, we say

$$h_1 \equiv h_2 \mod h^N$$

if $(h_1 - h_2)/h^N$ is analytic at all the points P where both h is equal to zero at P and both h_1 and h_2 are analytic at P. We will use a similar terminology for vector fields on $\mathcal{M}$.

Suppose we are given derivations $D_1, \dots, D_n$ of $R[[\epsilon]]$ and we are given vector fields $\chi, \chi_1, \dots, \chi_n$ on a manifold $\mathcal{M}$. Suppose we are also given a function h on $\mathcal{M}$ which is killed by χ and all the χ_j. We further suppose that the f and g do not have poles along the set $\{h = 0\}$.

Definition 1.0.2. In the above situation, we say that $(f, g, h; \chi, \chi_1, \dots, \chi_n)$ form a representation of $D_1, \dots, D_n$ if

$$\chi_i(P(f, g)) \equiv D_{i,\chi,M}(P)(f, g) \mod h^N$$

for any $P \in R[[\epsilon]]$ and any positive integer N and M sufficiently large depending on N. We also assume that these equations are true with the convention that $D_0 = \partial$ and that $\chi_0 = \chi$.

Definition 1.0.3. We say that D_i is slow under the above representation if $D_i(P)$ is not in the ideal $(\epsilon) \subset R[[\epsilon]]$ for some $P \in R[[\epsilon]]$, but χ_i vanishes on the set $h = 0$.

We can construct representations using algebraic geometry. Let T be the disk in $\mathbf{C}$ and let $\pi : \mathcal{X} \to T$ be a smooth proper family of curves of genus n. We will suppose there are sections $P : T \to \mathcal{X}$, $Q : T \to \mathcal{X}$, $R : T \to \mathcal{X}$ so that for each $t \in T$, we have $P(t) + Q(t)$ is a divisor on $\mathcal{X}_t = \pi^{-1}(t)$ linearly equivalent to the divisor $2R(t)$. We will assume that $P(0) = Q(0) = R(0)$. Choosing a homology basis of $H_1(\mathcal{X}, \mathbf{Z})$, we can identify $\mathbf{C}^g \times T$ locally with the relative Jacobian of the family $\mathcal{X} \to T$. Suppose $\gamma \in H_1(\mathcal{X}, \mathbf{Z})$ is given. We will suppose there is function h on T so that

$$\int_{Q(t)}^{P(t)} \omega = h(t) \int_{\gamma} \omega \tag{5}$$

for all holomorphic one forms on $\mathcal{X}_t = \pi^{-1}(t)$. Geometrically, (5) says that under the Abel-Jacobi map which sends a piece of the curve to $\mathbf{C}^g$, the secant line from Q to P passes through γ. This is meaningful as long as $P(t)$, $Q(t)$ are all close together. Under our identification, a point $(v, t) \in \mathbf{C}^g \times T$ gives a line bundle $\mathcal{L}$ on $\mathcal{X}_t$. When $h(t) = 1/N$ for N an integer, we can introduce the functions $A_{\mathcal{L}}$ and $B_{\mathcal{L}}$ as being defined at points (v, t) with $h(t) = 1/N$. It is easy to see that there are meromorphic functions f and g which coincide with A and B, and these f, g and h and certain linear flows produce a representation of the Toda $\mathcal{D}_{T_k}$. Further, by studying the geometry of the situation, we can show that this representation is slow for $\mathcal{D}_{T_2} + 2\mathcal{D}_{T_1}$. This turns out to mean that f can be computed asymptotically up to an additive constant from g. Lemma 2.6.1 contains the crucial step. It says that if f is computed in terms of g up to order h^n and if the representation is slow, then we can compute f in terms of g up to an additive constant modulo h^{n+1}. This computation is just that there is a $P \in R_0[[\epsilon]]$ so that $f = P(g) + C$ and this P is the desired P of Theorem 1.0.1. This is a strange relation, since for any particular N, there's no relation between A and B. The subtlety here is that the geometric genus of the Bloch spectrum for a arbitrary A and B of periodicity N grows with N, but the geometric genus of the Bloch spectrum associated to the $A_{\mathcal{L}}$ and $B_{\mathcal{L}}$ above remains g, although the arithmetic genus does grow as we let $h(t) = 1/N$. In fact, $\mathcal{X}_t$ is the normalization of the Bloch spectrum, but the Bloch spectrum has many nodes which are resolved by the normalization. Further, the line bundle $\mathcal{L}$ on the normalization of the Bloch spectrum becomes a torsion free sheaf on the Bloch spectrum, which is not locally free at new nodes.

The basic problem turns out to be to construct lots of such representations. We look at families of curves inside $\mathbf{P}^1 \times \mathbf{P}^1$ of bidegree $(n+1, 2)$ with affine coordinates x and y. Intuitively, the condition (5) imposes $g-1$ conditions, so there should be lots of curves satisfying the condition (5). We investigate curves near the following curve C_0 defined by

$$0 = (y^2 - x)(x-1)\left(x - \frac{1}{2^2}\right)\left(x - \frac{1}{3^2}\right)\cdots\left(x - \frac{1}{n^2}\right).$$

Our object roughly is to show that the subset of curves satisfying (5) is smooth of codimension $g-1$ near C_0. Further, we let $\mathcal{L}_0$ be a line bundle on C_0 of degree n which has degree one on all the vertical components $\{0 = (x - 1/k^2)\}$ of C_0 and degree zero on the component $\{y^2 - x = 0\}$. Then we can explicitly calculate the functions f and g we are interested in when we deform the pair $(C_0, \mathcal{L}_0)$ in certain directions. For instance, when we deform one of the nodes of C_0 away, but still have the curves satisfying (5). This gives enough information to produce generic enough $g's$.

There are several technical problems in establishing our results. One is finding a suitable definition of generic. Another is that f is only determined up to a constant by g. This problem is overcome by a monodromy argument Lemma 3.7.4.

The sequence of commuting derivations $\mathbf{D}_{T_k}$ can be put in the context of Poisson brackets, so that we can consider algebraically the setup of Hamiltonian completely integrable systems with conserved quantities in involution with respect to a Poisson bracket and the associated flows from the conserved quantities. I learned about this type of construction from papers of E. Frenkel. Let

$$\hat{R} = \mathbf{C}[\ldots, \hat{a}_{-1}, \hat{a}_0, \hat{a}_1, \ldots, \hat{b}_{-1}, \hat{b}_0, \hat{b}_1, \ldots].$$

We say a monomial in the $\hat{a}_k$ and $\hat{b}_k$ has weight r if the sum of the subscripts of the $\hat{a}_k$ and $\hat{b}_l$ sum to r. So the monomial $\hat{a}_1\hat{a}_2\hat{b}_{-3}$ has weight 0. Let $I_k \subset \hat{R}$ be the $\mathbf{C}$ span of all the elements of weight k. Let M_N be the ideal of R generated by

$$\hat{a}_N, \hat{a}_{N+1}, \ldots, \hat{a}_{-N}, \hat{a}_{-N-1}, \ldots; \ldots, \hat{b}_N, \hat{b}_{N+1}, \ldots, \hat{b}_{-N}, \hat{b}_{-N-1}.$$

Let $\hat{I}_k$ be the completion of I_k with respect to subspaces $I_k \cap M_N$ as $N \to \infty$. Then

$$\mathcal{F} = \bigoplus_k \hat{I}_k$$

is called the Fourier ring. $\mathcal{F}$ is naturally a graded ring. We can construct a series of maps $f_n : R[[\epsilon]] \to \mathcal{F}[[\epsilon]]$ so that $f_n(v^{(0)}) = \hat{a}_n$ and $f_n(w^{(0)}) = \hat{b}_n$ and the f_n behave like Fourier coefficients, e.g.,

$$f_n(HK) = \sum_{k+l=n} f_k(H) f_l(K).$$

One can form an analogous ring $\mathcal{F}_0$ from the ring

$$\hat{R}_0 = \mathbf{C}[\ldots, \hat{b}_{-1}, \hat{b}_0, \hat{b}_1, \ldots].$$

One can then show that the Toda derivations on $R[[\epsilon]]$ induce derivations on $\mathcal{F}[[\epsilon]]$ which are compatible with the f_n. Now each Toda lattice can be put in a Poisson framework and we can make a formal version of these Poisson brackets to obtain a Poisson bracket on $\mathcal{F}[[\epsilon]]$. Further, the Toda flows on $\mathcal{F}[[\epsilon]]$ come from conserved quantities in $\mathcal{F}[[\epsilon]]$. Let $\hat{I}_P \subset \mathcal{F}[[\epsilon]]$ be the closure of the ideal generated by all the Fourier coefficients $f_n(v^{(0)} - P)$ for $n \neq 0$ and a certain Casimir. Then we can find an induced Dirac bracket on $\mathcal{F}[[\epsilon]]/\hat{\mathcal{I}}_P \simeq \mathcal{F}_0[[\epsilon]]$ so that the Toda derivations come by bracketing with conserved quantities. Further, we can find $\hat{\beta}_k \in \mathcal{F}[[\epsilon]]/(\hat{\mathcal{I}}_P)$ for $k \in \mathbf{Z}$ so that modulo ϵ the β_k generate $\mathcal{F}_0$ topologically and satisfy the defining relations of the Virasoro algebra modulo ϵ.

A similar construction of the deformation of KdV discovered by Frenkel and Reshetikhin [4] in terms of difference equations has been made by Frenkel [3]. I believe the techniques of this paper will produce many such deformations of KdV hierarchy as well as deformations of $\mathcal{W}$-algebras. [6] contains some very interesting work on deformations of KdV inspired by [5], which was produced independently of this paper.

2. Differential algebra

2.1

Let R be the ring of polynomials with complex coefficients with generators $v^{(i)}$ and $w^{(j)}$ where i and j run over the nonnegative integers,

$$R = \mathbf{C}[v^{(0)}, w^{(0)}, v^{(1)}, w^{(1)}, \ldots]$$

We introduce a $\mathbf{C}$ derivation ∂ by the formulas

$$\partial v^{(i)} = v^{(i+1)}$$

and

$$\partial w^{(i)} = w^{(i+1)}.$$

Then ∂ on any polynomial in R is defined by the Leibnitz rule. We have a subring $R_0 \subset R$ defined to be the ring generated by the $w^{(n)}$.

This ring R is considered to be the ring of translation invariant differential operators in two functions $f(x)$ and $g(x)$. An element of R can be regarded as such a differential operator by making the substitutions

$$v^{(n)} = \frac{\partial^n f(x)}{\partial x^n}$$

and

$$w^{(n)} = \frac{\partial^n g(x)}{\partial x^n}$$

so that ∂ just becomes $\frac{\partial}{\partial x.}$ If $P \in R$ and $f(x)$ and $g(x)$ are $\mathcal{C}^\infty$ functions of x, then we define

$$P(f,g)(x)$$

to be the result of making the above substitution. So for example, if $P = v^{(1)}w^{(2)}$, then

$$P(f,g)(x) = \frac{\partial f(x)}{\partial x}\frac{\partial^2 g(x)}{\partial x^2}.$$

If f and g depend on a auxiliary variable t, then we write $P(f,g)(x,t)$.

The ring R can be used to study systems of equations:

$$\frac{\partial f(x,t)}{\partial t} = P(f,g)(x,t)$$
$$\frac{\partial g(x,t)}{\partial t} = Q(f,g)(x,t),$$

where P and Q are elements of R. We can encode the pair (P,Q) by defining a derivation $D_{(P,Q)}$.

Definition 2.1.1. $D_{(P,Q)}$ is the derivation of R commuting with the derivation ∂ with the additional properties

$$D_{(P,Q)}(v^{(0)}) = P$$

and

$$D_{(P,Q)}(w^{(0)}) = Q.$$

Any derivation of R commuting with ∂ is of this form.

2.2

We will mostly be concerned with the ring $R[[\epsilon]]$. The elements of this ring are formal power series in ϵ so that the coefficients of ϵ^n are just elements of R. We extend ∂ to be a continuous derivation of $R[[\epsilon]]$ by taking $\partial\epsilon = 0$. We next introduce an important series of maps $E_k : R[[\epsilon]] \to R[[\epsilon]]$ by the formulas

$$E_k(P) = P + k\epsilon\partial P + \frac{k^2\epsilon^2\partial^2 P}{2!} + \frac{k^3\epsilon^3\partial^3 P}{3!} + \cdots .$$

Formally, we can write

$$E_k = \exp(k\epsilon\partial).$$

We have that

$$E_k E_j = E_{k+j}.$$

Note that $E_k(v^{(0)})$ is just the Taylor series for $f(x + k\epsilon)$

$$E_k(v^{(0)})(f(x), g(x)) = f(x) + \frac{k\epsilon\partial f(x)}{\partial x} + \cdots ,$$

when we make the substitution of $f(x)$ for $v^{(0)}$ described above. Note that if D is a continuous derivation of $R[[\epsilon]]$ commuting with ∂ and $D(\epsilon) = 0$, then D is uniquely specified by $D(v^{(0)})$ and $D(w^{(0)})$. Conversely, given F and G in $R[[\epsilon]]$, we can find a continuous derivation D commuting with ∂ and with $D(\epsilon) = 0$. Let us call such a derivation a tame derivation.

We will use the ring $R[[\epsilon]]$ to describe the asymptotic behavior of difference equations.

Definition 2.2.1. S_1 is the ring of polynomials in the variables

$$\ldots, X_{-1}, X_0, X_1, X_2, \ldots$$

and the variables

$$\ldots, Y_{-1}, Y_0, Y_1, Y_2, \ldots$$

.

In our context, a difference equation will be given by two polynomials P_1 and P_2 in the X_i and Y_j. In order to facilitate substitution, we will write

$$P_1(\ldots, a, \hat{b}, c, \ldots; \ldots, \alpha, \hat{\beta}, \gamma, \ldots)$$

to mean the result of substituting a for X_{-1}, b for X_0, c for X_1 and also substituting α for Y_{-1}, β for Y_0, etc. That is the $\hat{}$ is just to indicate the variable to be substituted for X_0 or Y_0. Let S_2 be the polynomial ring over $\mathbf{C}$ with variables $\dots, a_{-1}, a_0, a_1, \dots$ and $\dots, b_{-1}, b_0, b_1, \dots$. (Of course, R, S_1 and S_2 are all the same polynomial ring on a denumerable number of variables, but it is convenient to have different names the variables.) Given P_1 and P_2, we can define a derivation $\mathcal{D}_{P_1,P_2}$ of S_2 by

$$\mathcal{D}_{P_1,P_2}(a_n) = P_1(\dots, a_{n-1}, \hat{a}_n, a_{n+1}, \dots; \dots, b_{n-1}, \hat{b}_n, b_{n+1}, \dots)$$

and

$$\mathcal{D}_{P_1,P_2}(b_n) = P_2(\dots, a_{n-1}, \hat{a}_n, a_{n+1}, \dots; \dots, b_{n-1}, \hat{b}_n, b_{n+1}, \dots).$$

Let T be the automorphism of S_2 defined by $T(a_n) = a_{n+1}$ and $T(b_n) = b_{n+1}$. Then $\mathcal{D}_{P_1,P_2}$ is translation invariant in the sense that $\mathcal{D}_{P_1,P_2}$ commutes with T. Conversely, any derivation of S_2 commuting with T is of the form $\mathcal{D}_{P_1,P_2}$. Since the commutator of derivations is a derivation, this allows us to define the commutator of (P_1, P_2) with (Q_1, Q_2) by

$$\mathcal{D}_{[(P_1,P_2),(Q_1,Q_2)]} = [\mathcal{D}_{P_1,P_2}, \mathcal{D}_{Q_1,Q_2}].$$

We will now define a continuous derivation D_{P_1,P_2} of $R[[\epsilon]]$ by

$$D_{P_1,P_2}(v^{(0)}) = P_1(\dots, E_{-1}(v^{(0)}), \hat{E}_0(v^{(0)}), E_1(v^{(0)}), \dots; \dots, E_{-1}(w^{(0)}), \hat{E}_0(w^{(0)}), E_1(w^{(0)}), \dots) \tag{6}$$

and

$$D_{P_1,P_2}(w^{(0)}) = P_2(\dots, E_{-1}(v^{(0)}), \hat{E}_0(v^{(0)}), E_1(v^{(0)}), \dots; \dots, E_{-1}(w^{(0)}), \hat{E}_0(w^{(0)}), E_1(w^{(0)}), \dots). \tag{7}$$

We then define

$$D_{P_1,P_2}(v^{(n)}) = \partial^n D_{P_1,P_2}(v^{(0)})$$

and

$$D_{P_1,P_2}(w^{(n)}) = \partial^n D_{P_1,P_2}(w^{(0)})$$

and extend by the Leibnitz rule. Note that the commutator $[D_{P_1,P_2}, \partial]$ vanishes on the generators $v^{(n)}$ and $w^{(n)}$, so D_{P_1,P_2} commutes with ∂. It is an exercise in the chain rule that

$$[D_{(P_1,P_2)}, D_{(Q_1,Q_2)}] = D_{[(P_1,P_2),(Q_1,Q_2)]}.$$

2.3

Suppose that we have an element $P \in R$. Let $\mathcal{M}$ be an analytic manifold and let χ be an analytic vector field on $\mathcal{M}$ and let f and g be two meromorphic functions on $\mathcal{M}$. We define $P_\chi(f,g)$ to be $P(f,g,\chi f,\chi g,\chi^2 f,\chi^2 g,\dots)$. Next, suppose we have a function h on $\mathcal{M}$. For any given positive integer N, we can extend this definition to $P \in R[[\epsilon]]$ by

$$P_{\chi,N}\left(\sum_{n=0}^{\infty} P_n \epsilon^n\right)(f,g) = \sum_{n=0}^{N} h^n P_{n,\chi}(f,g).$$

If h_1 and h_2 are two functions on $\mathcal{M}$, we say

$$h_1 \equiv h_2 \mod h^N$$

if $(h_1 - h_2)/h^N$ is analytic on an open dense set of the set $\{h = 0\}$. We will use a similar terminology for vector fields on $\mathcal{M}$.

Suppose we are given tame derivations $D_1,\dots,D_n$ of $R[[\epsilon]]$ and we are given vector fields $\chi,\chi_1,\dots,\chi_n$. Suppose we are also given a function h on $\mathcal{M}$ which is killed by χ and all the χ_j. We further suppose that the f and g do not have poles along the set $\{h = 0\}$.

Definition 2.3.1. In the above situation, we say that $\rho = (f,g,h;\chi,\chi_1,\dots,\chi_n)$ is a representation of $D_1,\dots,D_n$ if

$$\chi_i(P(f,g)) \equiv D_{i,\chi,M}(P)(f,g) \mod h^N$$

for any $P \in R[[\epsilon]]$ and any positive integer N and M sufficiently large depending on N. We also assume that these equations are true with the convention that $D_0 = \partial$ and that $\chi_0 = \chi$. We define $\rho(P) = P(f,g)$. We will only use $\rho(P)$ in congruences modulo h^N, so in the context of congruence, the formal power series in h makes sense. Analogously, suppose that $\overline{D}_1,\dots,\overline{D}_n \in R_0[[\epsilon]]$. $\rho = (g,h;\chi,\chi_1,\dots,\chi_n)$ is a representation of $\overline{D}_1,\dots,\overline{D}_n$ if

$$\chi_i(P(g)) \equiv D_{i,\chi,M}(P)(g) \mod h^N$$

for any $P \in R_0[[\epsilon]]$ and any positive integer N and M sufficiently large depending on N.

In the Definition, it suffices to check the cases $P = v^{(0)}$ and $P = w^{(0)}$ to check the equality of the definition for all P, since both sides are derivations.

Definition 2.3.2. Suppose $D = \sum a_i D_i$is a linear combinations of the D_i. We say that D is slow under ρ if there is a $P \in R[[\epsilon]]$ so that $D(P)$ is not in the ideal $(\epsilon) \subset R[[\epsilon]]$, but $\sum a_i \chi_i$ vanishes on the set $h = 0$.

2.4

We will be constructing representations be in the following context: Let V be an analytic manifold and let $\mathcal{M} = V \times \mathbf{C}^g$. Let π be the projection of $\mathcal{M}$ onto V. h will be the pullback of some function on V via π. $\sigma = \sigma_0$ and $\sigma_1, \ldots, \sigma_k$ will denote sections of π, $\sigma_k : V \to \mathcal{M}$. Now any section τ of π induces vertical vector field D_τ on $\mathcal{M}$ by

$$\mathbf{D}_\tau(f)(x) = \lim_{p \to 0} \frac{f(x + p\tau(x)) - f(x)}{p}.$$

We can define representations of $R[[\epsilon]]$ when we have $\mathcal{C}^\infty$ functions f and g which satisfy difference equations.

Proposition 2.4.1. *Suppose that f and g satisfy the following equations:*

(8)

$$\begin{aligned}\mathbf{D}_{\sigma_n}(f)(x) = P_{1,n}(\ldots, f(x - h(x)\sigma(\pi(x))), \hat{f}(x), f(x + h(x)\sigma(\pi(x))), \ldots;\\ \ldots, g(x - h(x)\sigma(\pi(x))), \hat{g}(x), g(x + h(x)\sigma(\pi(x))), \ldots)\end{aligned}$$

(9)

$$\begin{aligned}\mathbf{D}_{\sigma_n}(g)(x) = P_{2,n}(\ldots, f(x - h(x)\sigma(\pi(x))), \hat{f}(x), f(x + h(x)\sigma(\pi(x))), \ldots;\\ \ldots, g(x - h(x)\sigma(\pi(x))), \hat{g}(x), g(x + h(x)\sigma(\pi(x))), \ldots),\end{aligned}$$

where $P_{l,k}$ are in S_1 and the $\hat{}$ is the place indicator. We let D_k be the element $D_{P_{1,k},P_{2,k}}$, a tame derivation of $R[[\epsilon]]$ defined above and let $\chi_k = \mathbf{D}_{\sigma_k}$ and χ be $\mathbf{D}_\sigma$. Then $(f, g, h; \chi, \chi_1, \ldots, \chi_n)$ form a representation of $D_1, \ldots, D_n$.

Proof. All we need to do is to check that

$$\chi_i(f) \equiv D_{i,\chi,N}(v^{(0)})(f, g) \mod h^N$$

and

$$\chi_i(g) \equiv D_{i,\chi,N}(w^{(0)})(f, g) \mod h^N.$$

In fact,

(10)

$$
\begin{aligned}
&D_{i,\chi,N}(v^{(0)})(f,g) \\
&\equiv P_{1,n}(\ldots, f(x-h(x)\sigma(\pi(x))), \hat{f}(x), f(x+h(x)\sigma(\pi(x))), \ldots; \\
&\quad \ldots, g(x-h(x)\sigma(\pi(x))), \hat{g}(x), g(x+h(x)\sigma(\pi(x))), \ldots) \mod h^N.
\end{aligned}
$$

This in turn follows from

$$f(x+ph(x)\sigma(\pi(x))) \equiv E_p(v^{(0)})_{\chi,N}(f,g) \mod h^N,$$

which in turn is just Taylor's theorem. q.e.d.

2.5

We will frequently use this construction when

$$f = -2 + h^2 f_1$$

and

$$g = 1 + h^2 g_1,$$

where f_1 and g_1 are meromorphic functions of $\mathcal{M}$ which do not have polar divisors containing $\{h = 0\}$. To this end, define a $\mathbf{C}$ algebra endomorphism of $R[[\epsilon]]$ commuting with ∂ by

$$\Phi(v^{(0)}) = -2 + \epsilon^2 v^{(0)}$$

and

$$\Phi(w^{(0)}) = 1 + \epsilon^2 w^{(0)}.$$

Φ does not have an inverse on $R[[\epsilon]]$, but does have one on $R((\epsilon))$, the ring of Laurent series in ϵ which contain only finitely many negative powers of ϵ.

Definition 2.5.1. Suppose D is a tame derivation of $R[[\epsilon]]$. We define a new derivation D_Φ of $R((\epsilon))$ by

$$D_\Phi = \Phi D \Phi^{-1}.$$

In situations we will be considering D_Φ will turn out to be a tame derivation of $R[[\epsilon]]$.

Lemma 2.5.2. *Suppose $f = -2 + h^2 f_1$ and $g = 1 + h^2 g_1$. If*

$$\rho = (f, g, h; \chi, \chi_1, \dots, \chi_n)$$

is a representation of $D_1, \dots, D_n$ then $\rho_\Phi = (f_1, g_1, h; \chi, \chi_1, \dots, \chi_n)$ is a representation of $D_{1,\Phi}, \dots, D_{n,\Phi}$.

Proof.

$$(\Phi(P))(f_1, g_1) = P(f, g)$$

for any $P \in R[[\epsilon]]$. So

$$\begin{aligned} \chi_i((\Phi(P))(f_1, g_1)) &= \chi_i(P(f, g)) \\ &\equiv D_i(P)(f, g) \mod h^N \\ &\equiv \Phi(D_i(P))(f_1, g_1) \mod h^N \\ &\equiv (D_{\Phi,i})(\Phi(P))(f_1, g_1) \mod h^N. \end{aligned} \tag{11}$$

Given $Q \in R[[\epsilon]]$, we let $P = \Phi^{-1}(Q)$ and then we have

$$\chi_i(Q)(f_1, g_1)) \equiv (D_{\Phi,i})(Q)(f_1, g_1) \mod h^N$$

so we have a representation. q.e.d.

Next we work out a simple example of these definitions. We take

$$P_1 = Y_{-1} - Y_0 \in S_1$$

and

$$P_2 = Y_0(X_0 - X_1).$$

These are the Toda equations. Then define D_1 by

$$\begin{aligned} D_1(v^{(0)}) &= w^{(0)} - w^{(1)}\epsilon + w^{(2)}\epsilon^2/2! + \cdots - w^{(0)} \\ &= -w^{(1)}\epsilon + w^{(2)}\epsilon^2/2! + \cdots \end{aligned} \tag{12}$$

and

$$D_1(w^{(0)}) = w^{(0)}(-v^{(1)} - v^{(2)}\epsilon/2! + \cdots).$$

Then

$$D_{1,\chi,N}(v^{(0)})(f, g) = -h\chi(g) + h^2\chi^2(g)/2! + \cdots$$

and

$$D_{1,\chi,N}(w^{(0)})(f, g) = g(-h\chi(f) - h^2\chi^2(f)/2! + \cdots).$$

So if $(f, g, h; \chi_1)$ is a representation of D_1, then we will have

$$\chi_1(f) = -h\chi(g) + h^2\chi^2(g)/2! + \cdots$$

and

$$\chi_2(g) = g(-h\chi(f) - h^2\chi^2(f)/2! + \cdots),$$

where these equations are taken to be true near $\{h = 0\}$ modulo high powers of h, so the expansions are considered to be asymptotic.

2.6

Recall that $R_0 \subset R$ is the $\mathbf{C}$ algebra generated by the $w^{(n)}$. Next, we consider $P \in R_0[[\epsilon]]$ and assume that

$$P = w^{(0)} + \epsilon P_1.$$

Assume we have a representation $(f, g, h; \chi, \chi_1)$ of D_1. Assume that

$$D_1(v^{(0)}) = v^{(1)} - w^{(1)} + \epsilon Q_1$$

and that

$$D_1(w^{(0)}) = w^{(1)} - v^{(1)} + \epsilon Q_2.$$

Assume that this representation is slow for D_1, so that χ_1 vanishes on $h = 0$. Let $\mathcal{D}$ be the derivation defined by

$$\mathcal{D}(v^{(0)}) = v^{(1)} - w^{(1)}$$

and

$$\mathcal{D}(w^{(0)}) = w^{(1)} - v^{(1)}.$$

Then

$$D_1 = \mathcal{D} + \epsilon\mathcal{E}.$$

Let $W = (v^{(0)} - P)$ and let $\mathcal{I}_P$ be the closure of the ideal generated by W, ∂W, $\partial^2 W, \ldots$. If $Q \in R[[\epsilon]]$, let $\overline{Q}$ be the image of Q in $R[[\epsilon]]/\mathcal{I}_P$. Then there is the natural map

$$\sigma_1 : R_0[[\epsilon]] \to R[[\epsilon]]/\mathcal{I}_P.$$

Note that σ_1 is an isomorphism. We let $\sigma(Q)$ be $\sigma_1^{-1}(\overline{Q})$. Thus $\sigma(Q)$ is just the result of replacing any occurrence of $v^{(0)}$ in Q by P, any occurrence of $v^{(1)}$ by ∂P, etc.

Lemma 2.6.1. *Given P and D_1 as above, there is an $\mathcal{H} \in R_0[[\epsilon]]$ so that if*

$$\rho(v^{(0)} - P) \equiv 0 \mod h^n.$$

Then

$$\chi(\rho(v^{(0)} - P)) \equiv \rho(\mathcal{H}) \mod h^{n+1}.$$

Note that $\mathcal{H}$ does not depend on ρ.

Proof. We let $s = \rho(v^{(0)} - P)$. Notice that if $Q \in R$, then

$$\rho(Q) \equiv \rho(\sigma(Q)) \mod h^n, \tag{13}$$

since $P = \sigma(v^{(0)})$ and

$$\begin{aligned} \rho(v^{(0)}) &\equiv \rho(P) \mod h^n \\ &\equiv \rho(\sigma(v^{(0)})) \mod h^n. \end{aligned} \tag{14}$$

Let $x = \chi_1(s)$. Then

(15)

$$\begin{aligned} x &= \rho(D_1(v^{(0)} - P)) \\ &= \rho(D_1(v^{(0)} - w^{(0)} - \epsilon P_1)) \\ &= \rho(\mathcal{D}(v^{(0)} - w^{(0)} - \epsilon P_1) + \epsilon\mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)) \\ &= \rho(2\partial(v^{(0)} - w^{(0)}) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon(P_1)) \\ &= 2\chi(\rho(v^{(0)}) + \rho(-w^{(0)})) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)) \\ &= 2\chi(s) + 2\chi\rho(P) + \rho(-2w^{(1)}) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)) \\ &= 2\chi(s) + \rho(2\partial P - 2w^{(1)}) + \epsilon(\rho(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)). \end{aligned}$$

Noting that $\rho(\sigma(-\mathcal{D}(P_1)+\mathcal{E}(v^{(0)}-w^{(0)}-\epsilon P_1)))$ is congruent to $\rho(-\mathcal{D}(P_1) +\mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1))$ modulo h^n from (13), we can continue

$$\begin{aligned} x = 2\chi(s) + \rho(\sigma(2\partial P - 2w^{(1)} \\ + \epsilon(-\mathcal{D}(P_1) + \mathcal{E}(v^{(0)} - w^{(0)} - \epsilon P_1)) \mod h^{n+1}. \end{aligned} \tag{16}$$

But

$$x \equiv 0 \mod h^{n+1}$$

since D_1 is slow for ρ so we obtain the conclusion of the lemma. q.e.d.

Definition 2.6.2. Let $D_i(v^{(0)}) = P_i$ and $D_i(w^{(0)}) = Q_i$. We say $D_1, \ldots, D_n$ are A-nice if

$$P_i\left(v^{(0)} + K, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots\right)$$
$$= \sum_{j \le i} \binom{i}{j} P_j\left(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots\right)(AK)^j$$

and

$$Q_i\left(v^{(0)} + K, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots\right)$$
$$= \sum_{j \le i} \binom{i}{j} Q_j\left(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots\right)(AK)^j.$$

Definition 2.6.3. Let $D_i(v^{(0)}) = P_i$ and $D_i(w^{(0)}) = Q_i$. We say the ρ has weights $r_1, \ldots, r_n$ if

$$P_i\left(uv^{(0)}, uv^{(1)}, \ldots; u^2w^{(0)}, u^2w^{(1)}, \ldots\right)$$
$$= u^{r_i} P_i\left(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots\right).$$

and

$$Q_i\left(uv^{(0)}, uv^{(1)}, \ldots; u^2w^{(0)}, u^2w^{(1)}, \ldots\right)$$
$$= u^{r_i+1} Q_i\left(v^{(0)}, v^{(1)}, \ldots; w^{(0)}, w^{(1)}, \ldots\right).$$

Remark 2.6.4. Note that if the D_i are 1-nice, then $D_{i,\Phi}$ are ϵ^2-nice.

Definition 2.6.5. Suppose that $\rho = (f, g, h, \chi; \chi_1, \ldots, \chi_n)$ is a ϵ^2-nice representation and that s is a meromorphic function on $\mathcal{M}$ with $\chi(s) = 0$ and $\chi_i(s) = 0$. Let

$$\chi'_i = \sum_{j \le i} \binom{i}{j} \epsilon^{2j} \chi_j s^j$$

Then $\rho' = (f + s, g, h, \chi; \chi'_1, \ldots, \chi'_n)$ is a representation of $D_1, \ldots, D_n$ and we call ρ' the translation of ρ by s.

Definition 2.6.6. Suppose $\rho = (f, g, h, \chi; \chi_1, \ldots, \chi_n)$ has weights $r_1, \ldots, r_n$. We define an extended representation $\rho' = (f', g', h', \chi', \chi'_1, \ldots)$ on $\mathcal{M} \times \mathbf{C}$ by first defining a function $u : \mathcal{M} \times \mathbf{C} \to \mathbf{C}$ by

$$u(m, z) = 1 + h(m)^2 z$$

and

$$f'(m,z) = u(m,z)f(m)$$

and

$$g'(m,z) = u(m,z)^2 g(m)$$

and χ_i' is u^{1-r_i} times the natural pullback of χ_i and h' and χ' are the natural pullbacks of h and χ to $\mathcal{M} \times \mathbf{C}$.

Remark 2.6.7. This definition works fine without the particular choice of $u = 1 + h^2 z$ we have made, but the next remark does not.

Remark 2.6.8. In the situation of Definition 2.6.5, suppose that

$$f = -2 + h^2 f_1$$

and

$$g = 1 + h^2 g_1.$$

Then

$$\rho_\Phi'(w^{(0)})(m,z) \equiv \rho_\Phi(w^{(0)})(m) + 2z \mod h.$$

Definition 2.6.9. If $W \in R_0[[\epsilon]]$, we say $g = \rho(w^{(0)})$ satisfies the equation W nontrivially $\mod h^n$ if

$$\rho(W) \equiv 0 \mod h^n$$

and g does not vanish at a generic point of $\{h = 0\}$ and $W \notin (\epsilon^n)$.

Remark 2.6.10. Note that in the above definition, we can find $W_1 \in R_0[[\epsilon]]$ so that $W_1 = \epsilon^k W$ and $W_1 \notin (\epsilon)$. Then $k < n$, so $\rho(W_1)(g)$ is zero when restricted to $\{h = 0\}$.

2.7

Let $D \in R_0$. We introduce the variational derivative of D

$$\delta D = \sum_{k=0}^{\infty} (-1)^k \partial^k \left(\frac{\partial D}{\partial w_k} \right).$$

This operator has the property that

$$\left[\frac{d}{d\epsilon} \int_0^{2\pi} D(f + \epsilon g) \right]_{\epsilon=0} = \int_0^{2\pi} \delta D(f) g, \tag{17}$$

if f and g are periodic. $\delta(D)(f)$ can be somewhat more intuitively defined by Equation (17). Evaluating

$$\int_0^{2\pi} D(f+\epsilon g)$$

will yield the integral of a differential polynomial in f and g. To take the limit at $\epsilon \to 0$, we can throw away all the non-linear terms in g. Further, we can eliminate any occurences of derivatives of g by integration by parts. The resulting differential polynomial in f is $\delta(D)(f)$.

Lemma 2.7.1. *Suppose we have a function $f(t_1, t_2, \ldots, t_n; x)$ which is periodic with period 1 in x. Suppose that for generic z, we have*

$$\int_z^{z+1} D(f)(x)dx = 0.$$

Then

$$\int_z^{z+1} \delta(D)(f)\frac{\partial f}{\partial t_i} = 0,$$

for all i.

Proof. We assume that $i = 1$ for convenience. Fixing $t_1, t_2, \ldots$, we let

$$g(x,\epsilon) = \frac{f(t_1+\epsilon, t_2, \ldots, t_n) - f(t_1, t_2, \ldots, t_n)}{\epsilon}$$

Note that $g(x, \epsilon)$ is holomorphic function of x, even when $\epsilon = 0$. In fact,

$$g(x,0) = \frac{\partial f}{\partial x_1}.$$

So

$$\begin{aligned} \int_z^{z+1} D(f+\epsilon g) &= \int_z^{z+1} D(f)(t_1+\epsilon, t_2, \ldots, t_n; x)dx \\ &= 0. \end{aligned} \tag{18}$$

Taking the derivative of both sides of the equation with respect to ϵ and setting $\epsilon = 0$, we obtain

$$\int_z^{z+1} \delta(D)(f)\frac{\partial f}{\partial t_1} = 0.$$

q.e.d.

Lemma 2.7.2. *Let $D \in R_0$ be of order m, i.e., the highest derivative occuring is of order m. Suppose that*

$$\frac{\partial f}{\partial t_k}(0,0,\ldots,0;x) = a_k + b_k \exp(2\pi i k x),$$

with $b_k \neq 0$. Then $D(f)$ is not identically zero for $k \leq m$.

Proof. Fix a point x_0. We define a map $\phi_{x_0} : \mathbf{C}^m \to \mathbf{C}^m$ by

$$\phi_{x_0}\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ \vdots \\ t_m \end{pmatrix} = \begin{pmatrix} f(t_1,t_2,\ldots,t_m;x_0) \\ \frac{\partial f(t_1,t_2,\ldots,t_m;x_0)}{\partial x} \\ \frac{\partial^2 f(t_1,t_2,\ldots,t_m;x_0)}{\partial^2 x} \\ \vdots \\ \frac{\partial^m f(t_1,t_2,\ldots,t_m;x_0)}{\partial^m x} \end{pmatrix}.$$

We can compute the Jacobian matrix $d\phi_{x_0}$ of ϕ_{x_0} at $(0,0,\ldots,0)$.

$$\begin{aligned} & d\phi_{x_0}(0,0,0,\ldots,0) \\ &= \begin{pmatrix} a_1 + b_1 e^{2\pi i x_0} & a_2 + b_2 e^{4\pi i x_0} & \ldots & a_m + b_m e^{2m\pi i x_0} \\ 2\pi i e^{2\pi i x_0} & 4\pi i b_2 e^{4\pi i x_0} & \ldots & 2\pi i m b_m e^{2m\pi i x_0} \\ \ldots & \ldots & \ddots & \ldots \\ (2\pi i)^m e^{2\pi i x_0} & b_2 (4\pi i)^m e^{4\pi i x_0} & \ldots & b_m (2\pi i m)^m e^{2m\pi i x_0} \end{pmatrix}. \end{aligned}$$

We claim that $\det(d\phi_{x_0}(0,0,0,\ldots,0)) \neq 0$ for generic values of x_0. It suffices to show that $\det(W_{x_0}) \neq 0$, where

$$W_{x_0} = \begin{pmatrix} \frac{a_1}{b_1} e^{-2\pi i x_0} + 1 & \frac{a_2}{b_2} e^{-4\pi i x_0} + 1 & \ldots & \frac{a_m}{b_m} e^{-2\pi i m x_0} + 1 \\ 1 & 2 & \ldots & m \\ 1 & 4 & \ldots & m^2 \\ \ldots & \ldots & \ddots & \ldots \\ 1 & 2^m & \ldots & m^m \end{pmatrix}.$$

In particular if $\det(W_{x_0}) = 0$, then we would have

$$\det \begin{pmatrix} \frac{a_1}{b_1} z + 1 & \frac{a_2}{b_2} z^2 + 1 & \ldots & \frac{a_m}{b_m} z^m + 1 \\ 1 & 2 & \ldots & m \\ 1 & 4 & \ldots & m^2 \\ \ldots & \ldots & \ddots & \ldots \\ 1 & 2^m & \ldots & m^m \end{pmatrix} = 0$$

for any value of z. In particular, taking $z = 0$, we would have that a Vandermond determinant was zero.

So the image of ϕ_{x_0} would contain an open set. On the other hand, the equation $D(f) = 0$ says that the numbers

$$f(t_1, t_2, \ldots, t_m; x_0), \frac{\partial f(t_1,t_2,\ldots,t_m;x_0)}{\partial x}, \frac{\partial^2 f(t_1,t_2,\ldots,t_m;x_0)}{\partial^2 x}, \ldots, \frac{\partial^m f(t_1,t_2,\ldots,t_m;x_0)}{\partial^m x}$$

satisfy a nontrivial algebraic equation $H = 0$ independent of $t_1, t_2, \ldots,$ t_m. But $\{H = 0\}$ cannot contain an open set. q.e.d.

2.8

We wish to study representations in the following somewhat degenerate context: We have a sequence of functions g_r on $\mathcal{M}_r = V_r \times \mathbf{C}$ and let π_r be the projection of $\mathcal{M}_r \to V_r$. We can write $g_r(v, z)$, where $v \in V_r$ and $z \in \mathbf{C}$. We will assume that g_r are periodic with period one with respect to the second variable, e.g.,

$$g_r(v, z) = g_r(v, z + 1).$$

Let $\mathbf{D}_r$ be the vector field $\frac{\partial}{\partial z}$ which is tangent to the fibers of π_r. We can define $P(g_r)$ for all $P \in R_0$ so that $(w^{(0)})(g_r) = g_r$ and

$$\mathbf{D}_r(P(g_r)) = \partial(P)(g_r).$$

We also assume there are maps $k_r : V_r \to V_{r+1}$ so that the pullback of g_{r+1} is g_r via $k_r \times id$. We also assume we are given points $Q_r \in V_r$ compatible with the maps π_r.

Definition 2.8.1. We say the family $\{g_r\}$ is generic if first for any $D \in R_0$, there is an r so that $D(g_r)$ is not identically zero. Second suppose that we are given a $D \in R_0$. Each point v of V_r yields a function of x by

$$F_{v,r}(x) = g_r(v, x).$$

Suppose that

$$\int_z^{z+1} D(F_{v,r})(x)\, dx = 0$$

for generic z,v and all r. Then there is an $E \in R_0$ so that $D = \partial E$.

Next we give a criterion for the family $\{g_r\}$ to be generic. We further assume that for any positive integer n, there is an integer r so that for any integer k, $-n \leq k \leq n$ there are maps $\phi_{k,r}$ of the unit disk $D \subset \mathbf{C}$

to V_r with the following properties: Let $G_{r,k}$ be the pullback of g_r to $D \times \mathbf{C}$ via $\phi_{k,r} \times id$. Let $\mathbf{E}$ be the vector field on $D \times \mathbf{C}$ defined by

$$(\mathbf{E}G)(t,z) = \frac{\partial G(t,z)}{\partial t}.$$

We suppose that

$$(\mathbf{E}G_{r,k})(0,z) = a_{k,r} + b_{k,r}\exp(2\pi i k z), \tag{19}$$

where $a_{k,r}, b_{k,r} \in \mathbf{C}$ and $b_{k,r} \neq 0$ and $a_{r,0} = 0$. Further, we suppose that $\phi_{k,r}(0) = Q_r$.

$\phi_{k,r}$ induces a map $\psi_{k,r}$ from the tangent space T_0 of the disk D at zero to the tangent space T_{Q_r} of V_r. Note that the vectors

$$W_{k,r} = \psi_{k,r}\left(\frac{\partial}{\partial z}\right),$$

are all independent, since otherwise we would get a linear dependence relation between the functions $a_{k,r} + b_{k,r}\exp(2\pi i k z)$. We can introduce a coordinate system $t_1, t_2 \ldots t_p$ on V_r so that Q_r is the origin of the coordinate system and the span of the functions

$$\frac{\partial g_r}{\partial t_k}(0,0,0,\ldots,0;x)$$

contains the span of the functions $\exp(2\pi i k z)$ for k from $-m$ to m. This is because the linear span of the functions $\exp(2\pi i k z)$ is the same as the linear span of the functions $a_{k,r} + b_{k,r}\exp(2\pi i k z)$.

Lemma 2.8.2. *Suppose that for each r, we have that there is a periodic meromorphic function w_r on $V_r \times \mathbf{C}$ so that $\mathbf{D}_r(w_r) = D(g_r)$. Then under the assumption of (19), there is a $E \in R_0$ so that $D = \partial E$.*

Proof. If $\delta(D) = 0$, then we can find E so that $\partial E = D + C$, where C is constant. But our assumption implies that

$$\int_z^{z+1} D(g_r) = 0,$$

for generic z, where we integrate along a straight line for z to $z+1$ in $\mathbf{C}$. Since $E(g_r)$ is periodic, we have

$$\int_z^{z+1} \partial E(g_r) = 0.$$

So we would have $C = 0$. So we may assume that $\delta(D) \neq 0$. Note that $\delta(D) \notin \mathbf{C}$. Consequently, using Lemma 2.7.2 we can find a map $\gamma : D' \to V_m$ so that

$$(\delta(D)(g_r))(\gamma(t), x) = t^n G(t, x),$$

where $G(0, x)$ is not identically zero as a function of x and D' is the disk. Now $G(v, x + 1) = G(v, x)$. Thus for generic x there is an integer k so that

$$\int_x^{x+1} G(0, z) \exp(2\pi ikz) dz \neq 0.$$

Choose a p so that $\exp(2\pi ikz)$ is in the span of the functions

$$\frac{\partial g_p}{\partial t_l}(0, 0, 0, \dots, 0; x).$$

On the other hand, we have that

$$0 = \int_z^{z+1} \delta(D)(g_p) \frac{\partial g_p}{\partial t_l}$$

from Lemma 2.7.1 Hence we can find a function R so that

$$0 = \int_z^{z+1} \delta(D)(g_p)(\gamma(t), x) R(t, x)\, dx$$

and

$$R(Q_p, x) = \exp(2\pi ikz)$$

by taking R to be a linear combination of the functions $\frac{\partial g_p}{\partial t_l}$. In particular, we obtain

$$0 = \int_z^{z+1} \delta(D)(g_p)(\gamma(t), x)) R(t, x) \tag{20}$$
$$= t^n \int_z^{z+1} G(t, x) R(t, x).$$

So

$$0 = \int_z^{z+1} G(t, x) R(t, x).$$

Evaluating at $t = 0$ yields a contradiction. q.e.d.

Lemma 2.8.3. *Suppose that $D_1 \dots D_n \in R_0$. and that the family $\{g_r\}$ is generic. Then we can find an r so that if D is a nontrivial linear combination of the D_k and $D(g_r) = 0$, then $D = 0$.*

Proof. We can assume that the D_i are linearly independent. Let $W_r \subset \mathbf{C}^n$ be the set of $(a_1, \dots, a_n)$ so that

$$\sum_k a_k D_k(g_r) = 0.$$

W_r is a sequence of linear subspaces which decrease with r. Further, by the first property of generic $\{g_r\}$, any element of $\mathbf{C}^n$ is eventually not in some W_r. q.e.d.

2.9

We will next study the following situation: Let $\mathcal{N}_r = V_r \times \mathbf{C}^r$, where $\mathbf{C}^r$ has a basis $\delta_1, \dots, \delta_r$ and let π_r be the projection from $\mathcal{N}_r$ to V_r. We suppose there are inclusions $k_r : V_r \to V_{r+1}$. For simplicity, we will identify the image of V_r under k_r with V_r. Suppose there are functions h_r on V_r so that the restriction of h_{r+1} is h_r. Let $\mathcal{Q}_r$ be a series of points in V_r with $\mathcal{Q}_r = \mathcal{Q}_{r+1}$ under the identification and assume $h_r(\mathcal{Q}_r) = 0$. Then we can send $\psi_r : \pi_{r+1}^{-1}(V_r) \to \mathcal{N}_r$ by

$$\psi_r(v_r, c_1, \dots, c_{r+1}) = (v_r, c_1, \dots, c_r).$$

Suppose that f_r and g_r are functions on $\mathcal{N}_r$ and that $\psi_r^*(f_r)$ is the restriction of f_{r+1} and $\psi_r^*(g_r)$ is the restriction of g_{r+1}. We will assume that f_r and g_r are generically defined on the zero section on $\mathcal{N}_r \to V_r$.

Suppose that $D_1, \dots, D_n$ are tame derivations of $R[[\epsilon]]$ and that $\chi_{0,r}$, $\chi_{1,r}, \dots, \chi_{n,r}$ are vertical vector fields on $\mathcal{N}_r$ which form representations of $D_1, \dots, D_n$ and so that $\chi_{0,r}$ represents ∂. We will also assume that $\chi_{0,r}$ is differentiation in the direction $\delta_1 + 2\delta_r + \dots + r\delta_r$. Note that f_{r+1} and g_{r+1} are constant when restricted to the fibers of ψ_r and further we assume that if F is a function on $\mathcal{N}_r$,

$$\chi_{k,r+1}(\psi_r^*(F)) = \psi_r^*(\chi_{k,r}(F))$$

for $k \le r$.

Definition 2.9.1. We say that a sequence of representations as above are a compatible sequence.

Let $W_r \subset \mathcal{N}_r$ be defined by $h_r = 0$. We can generate a certain class of functions on W_r in the following way: Take any $P \in R[\epsilon]$. Then $P(f, g)$ is a meromorphic function on $\mathcal{N}_r$. Then we find the minimal power p of h_r so that $P(f, g)h_r^p$ is generically defined on W_r. We can then restrict $P(f, g)h_r^p$ to W_r to obtain a meromorphic function on W_r. Let $\mathcal{C}$ be the set of all functions on W_r arising from such P.

Definition 2.9.2. The sequence of compatible representations is general if the following holds: Suppose f is in the class $\mathcal{C}$. Suppose is that

$$f_1(v_r, z) = f(v_r, z, 2z, \dots, rz)$$

is constant as a function of z and $v_r \in V_r$ is generic in V_r. Then f is constant on the fibers of the projection $W_r \to V_r$.

Now define functions f'_r and g'_r on $V_r \times \mathbf{C}$ by the formulas:

$$f'_r(v_r, z) = f_r(v_r, z, 2z, \dots, rz).$$

$$g'_r(v_r, z) = g_r(v_r, z, 2z, \dots, rz).$$

Notice that the restriction of g'_{r+1} to $V_r \times \mathbf{C}$ is g'_r. Let $\mathcal{G}_r$ be the function g'_r restricted to $h_r = 0$. We will assume that the $(\mathcal{G}_r, \mathcal{Q}_r)$ form a generic sequence. We also assume that all functions in the class $\mathcal{C}$ are periodic when translated by $\delta_1 + 2\delta_2 + r\delta_r$ and that the compatible sequence of representations is general.

Lemma 2.9.3. *Given an integer n, suppose there is a $P \in R_0[\epsilon]$ so that:*

1. $f_r \equiv P(g_r) \mod h_r^n$.
2. $D_1(v^{(0)}) \equiv v^{(1)} - w^{(1)} \mod \epsilon$.
3. $D_1(w^{(0)}) \equiv w^{(1)} - v^{(1)} \mod \epsilon$.
4. $\chi_{1,r}$ *vanishes on* $\{h_r = 0\}$.
5. $\chi_{0,r}, \dots$ *form a general representation of* $D_1, \dots$.
6. *This representation is slow for* D_1.

Then we can find a $Q \in R_0[\epsilon]$ congruent to $P \mod \epsilon^n$ so that

$$\frac{f_r - Q(g_r)}{h_r^n}$$

is constant on the fibers of the projection $W_r \to V_r$.

Proof. Using Lemma 2.6.1, we can find a $Q_1 \in R_0[\epsilon]$ so that

$$\chi(f_r) \equiv Q_1(g_r) \mod h_r^{n+1}.$$

In particular, we can restrict and get

$$\mathbf{D}_r(f'_r) = Q_1(g'_r).$$

Notice that

$$(\partial P - Q_1)(g'_r) \equiv 0 \mod h_r^n.$$

Since the g'_r are generic, we must have

$$\partial P \equiv Q_1 \mod \epsilon^n.$$

If not, we can suppose that

$$\partial P \equiv Q_1 \mod \epsilon^k$$

for some maximal $k < n$. Then let

$$E = \frac{\partial P - Q}{\epsilon^k}$$

and let $F \in R_0$ be the constant term of E as a power series in ϵ. Then $E(g'_r)$ vanishes on the set $h_r = 0$. Thus $F = 0$, contradicting the maximality of k.

So we can write

$$Q_1 \equiv \partial P + \epsilon^n Q_2 \mod \epsilon^{n+1}$$

for some $Q_2 \in R_0$. In particular,

$$h_r^n Q_2(g'_r) \equiv \frac{d}{dz}\left(f'_r(v_r, z) - P(g_r)(v_r, z)\right) \mod h_r^{n+1}.$$

Thus $Q_2(\mathcal{G}_r)$ is the derivative of a periodic function. Since the family $\mathcal{G}_r$ is general, we must have $Q_2 = \partial Q_3$ and hence we may set $Q = P + \epsilon^n Q_3$. Then $(f'_r - Q(g'_r))(v_r, z, 2z, \ldots, rz)$ is constant as a function of z. So

$$\frac{f_r - Q(g_r)}{h^n}$$

is constant on the fibers of the projection $W_r \to V_r$. q.e.d.

Proposition 2.9.4. *Under the hypotheses of Lemma 2.9.3, we can find functions F_r and vector fields $\chi'_1, \ldots, \chi'_n$ so that $F_r, g_r, h_r, \chi, \chi'_1, \ldots, \chi'_n$ form a representation of $D_1, \ldots, D_n$ and a $Q \in R_0[[\epsilon]]$ so that*

$$F_r \equiv Q(G_r) \mod h_r^{n+1}$$

and

$$Q \equiv P \mod \epsilon^n.$$

Further, $\chi'_{1,r}$ vanishes on $\{h_r = 0\}$.

Proof. Let $s_{n,r}$ be the restriction of $f_r - Q(g_r)$ to the zero section of $\mathcal{N}_r$ over V_r. We can think of $s_{n,r}$ as a function on V_r and hence we can think of $s_{n,r}$ as a function on $\mathcal{N}_r$. We define $F_r = f_r - s_{n,r}$. We can modify the $\chi_{k,r}$ to $\chi'_{k,r}$ by translation by $-s_{n,r}$ This new family of representations remains compatible. $\chi'_{1,r}$ still vanishes on $\{h_r = 0\}$. For

$$\chi_1 \equiv \chi'_1 \mod h_r,$$

since the difference $\chi_1 - \chi'_1$ is divisible by $s_{n,r}$ Note that we start out with $f_r \equiv g_r \mod h_r$. q.e.d.

Theorem 2.9.5. *Suppose there is a generic compatible family of representations with:*

1. *$\mathcal{G}_r, \mathcal{Q}_r$ general.*
2. *$\chi_{1,r}$ vanishes on $\{h_r = 0\}$.*
3. *$D_1(v^{(0)}) \equiv v^{(1)} - w^{(1)} \mod \epsilon$.*
4. *$D_1(w^{(0)}) \equiv w^{(1)} - v^{(1)} \mod \epsilon$.*

Then there is a $Q \in R_0[[\epsilon]]$ so that

$$D_i(v^{(0)} - Q) \in \mathcal{I}_Q$$

for all i.

Proof. Suppose we have constructed a Q_n so that

$$f_r \equiv Q_n(g_r) \mod h^n$$

for a given n and all r. By the Proposition 2.9.4, we can find

$$Q_{n+1} \equiv Q_n \mod \epsilon^n$$

and modify f_r so that

$$f_r \equiv Q_{n+1}(g_r) \mod h_r^{n+1}.$$

Notice that for any $P \in R[[\epsilon]]$, we have that

$$P(f_r, g_r) \equiv \sigma_{Q_{n+1}}(P)(g_r) \mod h_r^{n+1}.$$

Since $\sigma_{Q_{n+1}}$ is a ring homomorphism commuting with ∂, it suffices to check the equation for $P = v^{(0)}$ and $P = w^{(0)}$. Now

$$\begin{aligned} 0 &\equiv \chi_{i,r}(f_r - Q_{n+1}(g_r)) \\ &\equiv D_i(v^{(0)} - Q_{n+1})(f_r, g_r) \\ &\equiv \sigma_{Q_{n+1}}(D_i(v^{(0)} - Q_{n+1}))(g_r) \mod h_r^{n+1}. \end{aligned} \tag{21}$$

We let Q be the limit of the Q_n. Then

$$\sigma_Q(D_i(v^{(0)} - Q))(g_r) \equiv 0 \mod h_r^n$$

for all n. Since the g_r are generic, we have

$$\sigma_Q(D_i(v^{(0)} - Q)) = 0,$$

i.e., $D_i(v^{(0)} - Q) \in \mathcal{I}_Q$. q.e.d.

Definition 2.9.6. Suppose that

$$D_i(v^{(0)} - Q) \in \mathcal{I}_Q.$$

Then we define derivations

$$\overline{D}_i : R_0[[\epsilon]] \to R_0[[\epsilon]]$$

by

$$\overline{D}_i(P) = \sigma(\overline{D_i(P)})$$

where σ is the inverse of the natural map

$$\sigma_1 : R_0[[\epsilon]] \to R[[\epsilon]]/\mathcal{I}_Q$$

and $\overline{T}$ is the image of $T \in R[[\epsilon]]$ in $R[[\epsilon]]/\mathcal{I}_Q$.

Theorem 2.9.7. *Suppose that* $\rho = (f, g, h; \chi_1, \ldots, \chi_n)$ *is a representation of* $D_1, \ldots, D_n$ *and that*

$$f \equiv Q(g) \mod h^n$$

for all n. *Then using the notation of Definition 2.9.6,* $(g, h; \chi_1, \ldots, \chi_n)$ *is a representation of* $\overline{D}_1, \ldots, \overline{D}_n$.

Proof.

$$\begin{aligned} \chi_i(g) &\equiv D_i(w^{(0)})(f, g)) \mod h^n \\ &\equiv D_i(w^{(0)})(Q(g), g) \mod h^n \\ &\equiv \sigma(D_i(w^{(0)})(g) \mod h^n \\ &\equiv \overline{D}_i(w^{(0)})(g) \mod h^n. \end{aligned} \tag{22}$$

q.e.d.

Definition 2.9.8. If $D \in R_0[[\epsilon]]$ so that

$$D = D_0 + D_1\epsilon + D_2\epsilon^2 + \cdots ,$$

we define

$$D^{[0]} = D_0.$$

Similarly, we let $\chi_i^{[0]}$ be the restriction of χ_i to the set $\{h = 0\}$.

Lemma 2.9.9. *Suppose that* $D_1, \ldots, D_n \in R_0$. *Suppose that* $E_1, \ldots, E_n \in R_0$. *Suppose that* $(g, 0; \chi_1, \ldots, \chi_n)$ *is a representation of* $D_1, \ldots, D_n$ *and* $(g, 0; \chi'_1, \ldots, \chi'_n)$ *is a representation of* $E_1, \ldots, E_n$. *Assume further that* $\chi_1, \ldots, \chi_n$ *and* $\chi'_1, \ldots, \chi'_n$ *span the same* n *dimensional vector space. Further, assume that if* g *satisfies any nontrivial linear combination* D *of the* D_i *and* E_i, *then* $D = 0$. *Then linear span of the* D_i *is the same as the linear span of the* E_i.

Proof. We can assume that the $\chi'_i = \chi_i$. Then $(D_i - E_i)(g) = 0$, so $D_i = E_i$. q.e.d.

Corollary 2.9.10. *Suppose that* $D_1 \in R_0$. *Suppose that* $(g, 0; \chi_1)$ *is a representation of* D_1. *Further, assume that* $D_1(g) = 0$ *implies that* $D_1 = 0$. *Then if* $\chi_1(g) = 0$, *then* $D_1 = 0$.

3. Constructing deformations

3.1

Let T be a smooth analytic manifold and let $\pi : \mathcal{X} \to T$ be a proper flat family of curves of arithmetic genus n. Let $t_0 \in T$ be a fixed point. We will usually be interested in the behavior of the family and related objects near t_0. We will suppose there are sections $P : T \to \mathcal{X}$, $Q : T \to \mathcal{X}$, $R : T \to \mathcal{X}$ so that for each $t \in T$, we have $P(t) + Q(t)$ is a divisor on $\mathcal{X}_t = \pi^{-1}(t)$ is linearly equivalent to the divisor $2R(t)$. We will also assume that $P(t)$, $Q(t)$ and $R(t)$ are all smooth points of $\mathcal{X}_t$.

Let $\delta : T \times S^1 \to \mathcal{X}$ be a map defined over T so that π is smooth at every point of the image of δ and let ω be a section of $\pi_*(\omega_{\mathcal{X}/T})$. ($\omega_{\mathcal{X}/T}$ is the sheaf of relative dualizing differentials and S^1 is the unit circle.) Then $\delta(t)$ is a cycle on $\mathcal{X}_t$ and we can form

$$I_\delta(t)(\omega_t) = \int_{\delta(t)} \omega_t$$

where I_δ is a section of the dual of $\pi_*(\omega_{\mathcal{X}/T})$. So we can consider I_δ as a section of $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$. We will suppose that we can find $\delta_1, \dots, \delta_n$ so that $I_{\delta_1}, \dots, I_{\delta_n}$ are a basis of $R^1\pi_*(\mathcal{O}_{\mathcal{X}})$. Let v_k be sections of $\pi_*(\omega_{\mathcal{X}/T})$ which are dual to the I_{δ_k}.

We will assume that there is an open set $U \subset \mathcal{X}$ so that $z : U \to D \times T$ is an isomorphism of $D \times T$ as T spaces, where D is the unit disk. We assume that the images of the sections P, Q and R are all contained in U. For simplicity, we will assume that the image of P is the zero section and let

$$h = z(R).$$

We will assume that

$$dz = v_1.$$

Further, we will assume that we can find anti-derivatives h_k of the v_k defined on U and that the functions

$$\frac{\int_Q^P v_k}{\int_Q^P v_1} = k.$$

This condition insures that the secant line between Q and P on the local curve

$$t \to (h_1(t), h_2(t), \dots, h_n(t))$$

passes through $(1, 2, \ldots, n)$ when $h_1(t) \neq 0$ and insures that the tangent line to P passes through $(1, 2, \ldots, n)$ when $h_1(t) = 0$. The existence of such families is not obvious at this point, but we will exhibit such families in §4.

3.2

Let

$$T_1 = T \times \mathbf{C}^n$$

and let

$$\mathcal{X}_1 = \mathcal{X} \times \mathbf{C}^n = \mathcal{X} \times_T T_1.$$

We will define a relative line bundle $\mathcal{L}$ on $\mathcal{X}_1$ over T_1. Let $\pi_1 : \mathcal{X}_1 \to T_1$ be the projection. We have the coordinate functions $z_1, \ldots, z_n$ on $\mathbf{C}^n$. We will again denote the pullback of z_i to T_1 by z_i. On the other hand, we will denote the pullback of I_{δ_i} to a section of $R^1\pi_{1,*}(\mathcal{O}_{\mathcal{X}_1})$ by I_{δ_i}. Thus we can consider $I = \sum z_k I_{\delta_k}$ as a section of $R^1\pi_{1,*}(\mathcal{O}_{\mathcal{X}_1})$. Then there is a relative line bundle $\mathcal{L}$ on $\mathcal{X}_1$ so that $\mathcal{L}$ corresponds to the same section of $R^1\pi_{1,*}(\mathcal{O}^*_{\mathcal{X}_1})$ as $\exp(2\pi i I)$. We also assume that we have a line bundle $\mathcal{M}$ of relative degree n on $\mathcal{X}_1$ which is the pullback of a line bundle on $\mathcal{X} \to T$. Let $\mathcal{N} = \mathcal{M} \otimes \mathcal{L}$.

We will assume that $P(T)$ and $Q(T)$, which are divisors on $\mathcal{X}$, meet transversally. Consider $Z = \pi(P(T) \cap Q(T))$, where π is the projection of $\mathcal{X} \to T$. We will assume Z is a divisor in T and that the map from $P(T) \cap Q(T) \to Z$ is an isomorphism. Let Z_1 be the inverse image of Z in T_1. We assume that $R(T) \cap P(T)$ and $R(T) \cap Q(T)$ are both equal to $P(T) \cap Q(T)$. We will assume that Z_1 is defined by an equation $h = 0$, where h is the pullback of a function on T.

We will also assume that we have chosen a nonconstant section λ of $\mathcal{O}_{\mathcal{X}}(P(T) + Q(T) - 2R(T))$. Let's fix a point $t \in T_1$. Then on $\mathcal{X}_{1,t}$, we have the line bundles $\mathcal{N}_{k,t} = \mathcal{N}_t(k(P(t) - Q(t)))$.

Definition 3.2.1. Suppose $P(t) \neq Q(t)$. Then t is N-good if each of the line bundles $\mathcal{N}_{k,t}(-P(t))$, is non-special for $|k| < N$. We then define sections $s_k \in H^0(\mathcal{N}_{k,t})$ for $|k| < N$ so that

$$\frac{s_{k+1}}{\lambda s_k}(P(t)) = 1.$$

Remark 3.2.2. Given our choice of λ, these s_k are determined up to an nonzero multiplicative constant independent of k.

We will assume we have made a special choice to λ to be denoted λ_0 so that $z\lambda_0(P) = 1$. Let's fix N and let $T_2 \subset T_1$ be the set of N-good points.

Theorem 3.2.3. *There are functions $A_{k,t,\lambda}$ and $B_{k,t,\lambda}$ defined on T_2 for $|k| < N - 1$ so that*

$$\lambda s_k = s_{k+1} + A_{k,t,\lambda} s_k + B_{k,t,\lambda} s_{k-1}.$$

The function B is never zero on T.

Proof. The dimension of $H^0(\mathcal{N}_p(P(p) + Q(p))) = 3$. So there must be a linear dependence relations

$$C\lambda s_k + D s_{k+1} + A s_k + B s_{k-1} = 0,$$

where A, B, C and D are all in $\mathbf{C}$. Now we have chosen our normalizations of the s_m and of λ so that

$$\frac{s_{k+1}}{\lambda s_k} = 1$$

at $P(p)$. We must then have $C = -D$. Further, we cannot have $C = D = 0$, since this would lead to a dependence relation between s_k and s_{k-1}, which have different order poles at $P(p)$. So we can completely normalize by taking $C = -1$ and $D = 1$ and finally writing

$$\lambda s_k = s_{k+1} + A s_k + B s_{k-1}.$$

q.e.d.

If u is a nowhere zero function on T, then

$$A_{k,t,u\lambda} = u A_{k,t,\lambda}$$

and

$$B_{k,t,u\lambda} = u^2 A_{k,t,\lambda}.$$

Further, we have

$$A_{k,t,C+\lambda} = A_{k,t,\lambda} + C,$$

while B is unchanged by adding C to λ for any function C on T. We can now normalize the B_k in the following way: Let

$$\mathbf{B}_{k,t} = \frac{B_{k,t,\lambda}}{B_{k,0,\lambda}}.$$

3.3

We next discuss theta functions following [8]. Let $C = \mathcal{X}_t$ and denote P_t by P, etc. We assume that C is nonsingular. We regard $H^1(C, \mathbf{Z})$ as a subgroup of $H^0(C, \Omega)^*$. Let $\mathbf{C}_1^*$ be the set of all complex numbers of absolute value one. Choose a map

$$\alpha : H^1(C, \mathbf{Z}) \to \mathbf{C}_1^*$$

so that

$$\frac{\alpha(u_1 + u_2)}{\alpha(u_1)\alpha(u_2)} = e^{i\pi\langle u_1, u_2\rangle}.$$

There is a unique hermitian form H on $H^0(C, \Omega)^*$ so that

$$\Im H(x, y) = \langle x, y\rangle.$$

Let ϑ defined on $H^0(C, \Omega)^*$ be the function satisfying the functional equation

$$\vartheta(z + u) = \alpha(u) e^{\pi H(z,u) + \pi H(u,u)/2} \vartheta(z)$$

for $z \in H^0(C, \Omega)^*$ and $u \in H^1(C, \mathbf{Z})$. There is a map $\gamma : U \cap C \to H^0(C, \Omega)^*$ defined by

$$\gamma(q) = \int_P^q \omega,$$

where the path from P to q is chosen to lie in $U \cap C$. Thus we can regard $\gamma(q) \in H^1(C, \mathcal{O})$. Given any non-special line bundle $\mathcal{L} = \mathcal{O}_C(D)$ of degree n on C with D effective, here is a constant $K_{\mathcal{L}} \in H^1(\mathcal{O}_C)$ so that the zeros of the function $p \to \vartheta(\gamma(p) + K_{\mathcal{L}})$ are just D counting multiplicity. Further,

$$\exp(2\pi i K_{\mathcal{L}}) = \mathcal{L}.$$

Also

$$K_{\mathcal{L}(q-P)} = K_{\mathcal{L}} - \gamma(q)$$

modulo periods. Let K_0 be a constant corresponding to the line bundle $\mathcal{M}$ and choose a line bundle $\mathcal{M}_1 = \mathcal{O}(D_1)$ so that P is a point of multiplicity one of D_1 and all the other points of D_1 are outside U. Select K_1 corresponding to $\mathcal{M}_1$. So $\vartheta(\gamma(z) + K_1)$ vanishes exactly once at P and at no other point of U.

Using the theta function, we can write down an expression for a function λ_0 initially valid in U.

$$\lambda_0(z) = \alpha \frac{\vartheta(\gamma(z) - \gamma(R) + K_1)^2}{\vartheta(\gamma(z) + K_1)\vartheta(\gamma(z) + K_1 - \gamma(Q))},$$

where

$$\alpha = \frac{\nabla(\vartheta)(K_1) \cdot \gamma'(P)\vartheta(K_1 - \gamma(Q))}{\vartheta(K_1 - \gamma(R))^2}.$$

This is a well-defined meromorphic function on C, since $P+Q$ is linearly equivalent to $2R$ so by Abel's theorem, $\gamma(P)+\gamma(Q) = 2\gamma(R)$. Using the periodicity properties of ϑ,

$$\frac{\vartheta(Z - \gamma(R) + K_1)^2}{\vartheta(Z + K_1)\vartheta(Z + K_1 - \gamma(Q))}$$

is periodic in $Z \in H^1(\mathcal{O})$.

Next, we develop a formula for

$$u = \frac{\lambda_0 s_0}{s_1}.$$

Attached to the point t, there is a line bundle $\mathcal{N}_{0,t}$ and let L be the projection on t to $\mathbf{C}^n$. The zeros of $\vartheta(\gamma(z) + L + \gamma(Q))$ match the zeros of s_1 and the zeros of $\vartheta(\gamma(z) + L)$ match the zeros of s_0. On the other hand,

$$\frac{\vartheta(\gamma(z) + K_1 - \gamma(Q))}{\vartheta(\gamma(z) + K_1)}$$

has a simple pole at P and a simple zero at Q. So s_1/s_0 is a multiple of the rational function

$$\frac{\vartheta(\gamma(z) + K_1 - \gamma(Q))\vartheta(\gamma(z) + L + \gamma(Q))}{\vartheta(\gamma(z) + K_1)\vartheta(\gamma(z) + L)}.$$

Consequently, u is a multiple of

$$\frac{\vartheta(\gamma(z) + K_1 - \gamma(R))^2\vartheta(\gamma(z) + L)}{\vartheta(\gamma(z) + K_1 - \gamma(Q))^2\vartheta(\gamma(z) + L + \gamma(Q))}.$$

Using that $u(P) = 1$ and $\gamma(P) = 0$ we obtain that

$$u = \frac{\vartheta(\gamma(z) + K_1 + \gamma(R))^2\vartheta(\gamma(z) + L)\vartheta(K_1 - \gamma(Q))^2\vartheta(L + \gamma(Q))}{\vartheta(\gamma(z) + K_1 - \gamma(Q))^2\vartheta(\gamma(z) + L + \gamma(Q))\vartheta(K_1 - \gamma(R))^2\vartheta(L)}.$$

Let a denote the constant term of the Laurent series for $\lambda - s_1/s_0$ developed around $z = 0$. We get $a = \frac{d}{dz}(u)$ evaluated at $z = P$. Since $u = 1$ at $z = 0$, we obtain

$$a = -2\frac{\nabla(\vartheta)(K_1 - \gamma(Q)) \cdot \gamma'(P)}{\vartheta(K_1 - \gamma(Q))} - \frac{\nabla(\vartheta)(L + \gamma(Q)) \cdot \gamma'(P)}{\vartheta(L + \gamma(Q))} + 2\frac{\nabla(\vartheta)(K_1 + \gamma(R)) \cdot \gamma'(P)}{\vartheta(K_1 + \gamma(R))} + \frac{\nabla(\vartheta)(L) \cdot \gamma'(P)}{\vartheta(L)}. \tag{22}$$

For future reference, we give a formula for the constant term C_0 of the Laurent expansion of s_1/s_0, namely

$$C_0 = \frac{\nabla(\vartheta)(K_1 - \gamma(Q)) \cdot \gamma'(P)}{\vartheta(K_1 - \gamma(Q))} + \frac{\nabla(\vartheta)(L + \gamma(Q)) \cdot \gamma'(P)}{\vartheta(L + \gamma(Q))} - \frac{\nabla(\vartheta)(L) \cdot \gamma'(P)}{\vartheta(L)} - \frac{W}{\nabla(\vartheta)(L) \cdot \gamma'(P)} \tag{23}$$

where

$$W = \frac{1}{2}\frac{d^2}{dz^2}(\vartheta(\gamma(z)))$$

evaluated at $z = P$.

We will also be interested in evaluating

$$v = \frac{\lambda_0 s_0}{s_{-1}}$$

at Q. We obtain that

$$\frac{s_{-1}}{s_0} = F\frac{\vartheta(\gamma(z) + K_1)\vartheta(\gamma(z) + L - \gamma(Q))}{\vartheta(\gamma(z) + K_1 - \gamma(Q))\vartheta(\gamma(z) + L)}$$

with

$$F = \frac{\vartheta(K_1 - \gamma(Q))\vartheta(L)}{\nabla(\vartheta)(K_1) \cdot \gamma'(P)\vartheta(L - \gamma(Q))}.$$

So

$$\frac{\lambda_0 s_0}{s_{-1}} = F_1\frac{\vartheta(\gamma(z) + L)\vartheta(\gamma(z) - \gamma(R) + K_1)^2}{\vartheta(\gamma(z) + K_1)^2\vartheta(\gamma(z) + L - \gamma(Q))},$$

where

$$F_1 = \frac{(\nabla(\vartheta)(K_1) \cdot \gamma'(P))^2\vartheta(L - \gamma(Q))}{\vartheta(-\gamma(R) + K_1)^2\vartheta(L)}.$$

So evaluating at Q, we get

$$\begin{aligned} b &= \frac{\lambda_0 s_0}{s_{-1}}(Q) \\ &= F_1\frac{\vartheta(L + \gamma(Q))\vartheta(\gamma(Q) - \gamma(R) + K_1)^2}{\vartheta(\gamma(Q) + K_1)^2\vartheta(L)} \\ &= \left(\frac{\vartheta(\gamma(Q) - \gamma(R) + K_1)\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\vartheta(-\gamma(R) + K_1)\vartheta(\gamma(Q) + K_1)}\right)^2 \\ &\quad \cdot \frac{\vartheta(\gamma(Q) + L)\vartheta(-\gamma(Q) + L)}{\vartheta(L)^2} \end{aligned} \tag{24}$$

$$= \left(\frac{\vartheta(\gamma(R) + K_1)\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\vartheta(-\gamma(R) + K_1)\vartheta(2\gamma(R) + K_1)} \right)^2 \cdot \frac{\vartheta(2\gamma(R) + L)\vartheta(-2\gamma(R) + L)}{\vartheta(L)^2}$$

bearing in mind that $2\gamma(R) = \gamma(Q)$.

We have defined a and b in Equations (22) and (24). a and b depend on t.

Proposition 3.3.1 *For our choice of λ_0, we have*

$$a = A_{0,t,\lambda_0}$$

and

$$b = B_{0,t,\lambda_0}$$

near t_0.

3.4

We will now make a canonical choice of λ near the set Z. Specifically, let

$$U = \sqrt{B_{0,(x,0),\lambda_0}}.$$

The expression for b makes it clear that such a meromorphic function exists, since

$$\lim_{\gamma(R)\to 0} \frac{\vartheta(L)^2}{\vartheta(2\gamma(R) + L)\vartheta(-2\gamma(R) + L)} = 1$$

for any L, in particular for $L = 0$. In particular, we set

$$U = \left(\frac{\vartheta(\gamma(R) + K_1)\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\vartheta(-\gamma(R) + K_1)\vartheta(2\gamma(R) + K_1)} \right) \sqrt{\frac{\vartheta(2\gamma(R))\vartheta(-2\gamma(R))}{\vartheta(0)^2}}$$

where we choose the branch of the square root which is near 1. When R is near P, then

$$\vartheta(\gamma(R) + K_1) \approx \gamma(R) \cdot \nabla(\vartheta)(K_1)$$

and

$$\vartheta(-\gamma(R) + K_1) \approx -\gamma(R) \cdot \nabla(\vartheta)(K_1)$$

and

$$\vartheta(2\gamma(R) + K_1) \approx 2\gamma(R) \cdot \nabla(\vartheta)(K_1)$$

so

$$U \approx -\frac{\nabla(\vartheta(K_1) \cdot \gamma'(P)}{\nabla(\vartheta)(K_1) \cdot \gamma(Q)}.$$

On the other hand, we have that

$$\vartheta(K_1 - \gamma(Q)) \approx -\nabla(\vartheta(K_1)) \cdot \gamma(Q),$$

so the constant term of s_1/s_0 is approximately

$$-\frac{\nabla(\vartheta)(K_1) \cdot \gamma'(P)}{\nabla(\vartheta)(K_1) \cdot \gamma(Q)}$$

from the expression for C_0. So the constant term of $U^{-1}s_1/s_0$ is just 1. Now choose C, a function on T, so that

$$U^{-1}A_{0,(x,0),\lambda_0} + C = -2.$$

Proposition 3.4.1. *We can canonically choose*

$$\lambda_{\text{can}} = U^{-1}\lambda_0 + C$$

so that:

1. $2 + A_{k,t,\lambda_{\text{can}}}$ *is divisible by* h^2.
2. $-1 + B_{k,t,\lambda_{\text{can}}}$ *is divisible by* h^2.
3. $2 + A_{k,0,\lambda_{\text{can}}} = 0$.
4. $-1 + B_{k,0,\lambda_{\text{can}}} = 0$.
5. *We have*
$$\lambda = \alpha_{-1}\frac{h}{z} + \alpha_0 + \alpha_1\frac{z}{h} + \cdots,$$
where $\alpha_{-1}(t)$ *vanishes on* Z *and* α_0 *becomes* -1 *on* Z.

Of course, the canonical choice depends on the line bundle $\mathcal{M}$, *but is otherwise completely determined near* Z *by properties three, four and five.*

Proof. Note that

$$V(z,h) = \lambda_0\frac{z(z-Q)}{(z-R)^2}$$

does not have zeros or poles when $h \neq 0$ and h and z are near zero. Now near $h = z = 0$, $V(z,h)$ is a power of h times a unit. We have

$$\lambda_{\mathrm{can}} = \frac{U^{-1}s_{n+1}}{s_n} + A_{\mathrm{can}} + B_{\mathrm{can}}\frac{Us_{n-1}}{s_n}$$

Hence the constant term of λ_{can} as a Laurent series in z is the constant term of

$$\frac{U^{-1}s_{n+1}}{s_n}$$

plus the constant term of A_{can}. So modulo h, the constant term of λ_{can} is -1. The Laurent series for λ_0 is

$$\frac{1}{z} + \frac{J}{h} + \cdots .$$

So $hV(c,h)$ is a unit. But the Laurent series of $hV(z,h)$ is a power series in z/h. So λ_{can} is a power series in $\frac{z}{h}$. q.e.d.

Definition 3.4.2.

$$\begin{aligned} \mathbf{A}_{k,t} &= A_{k,t,\lambda_{\mathrm{can}}} \\ \mathbf{f} &= \mathbf{A}_{0,t} \\ \mathbf{g} &= \mathbf{B}_{0,t}. \end{aligned}$$

$\mathbf{f}$ and $\mathbf{g}$ are functions on $T \times \mathbf{C}^n$.

Lemma 3.4.3. *$\mathbf{f}$ and $\mathbf{g}$ are defined near any 2-good point.*

Let χ be differentiation in the direction $(1, 2, \ldots, n)$.

Proposition 3.4.4.

$$\mathbf{g}(t,L) = 1 + 4h(t)^2(\chi^2(\log(\vartheta)(L) - \chi^2(\log(\vartheta)(0)))) + \textit{ higher order terms in } h.$$

Proof. We define

$$\mathbf{g}_1(L) = \frac{\vartheta(L)^2}{\vartheta(L+\gamma(Q))(\vartheta(L-\gamma(Q))}$$

and then

$$\mathbf{g} = \frac{\mathbf{g}_1(L)}{\mathbf{g}_1(0)}.$$

On the other hand,

$$\gamma(Q) = 2h(1, 2, \ldots, n)$$

and we know that $\gamma''(Q/2) = 0$. q.e.d.

Let's examine the function $\mathbf{G}(t, L)$ which is the analytic continuation of

$$\frac{\mathbf{g}(t, L) - 1}{h(t)^2}.$$

We can introduce a series of vectors $\mathbf{v}_k \in \mathbf{C}^n$ by using the identification of $\mathbf{C}^n$ with

$$H^0(\omega_{\mathcal{X}_{1,t}})^*$$

via integration over the δ_i by

$$\mathbf{v}_k(\omega) = \mathrm{Res}_{P(t)} \frac{\omega}{z^k}.$$

Let $\mathbf{K}_1, \mathbf{K}_2, \ldots$ be the KdV differential operators. Let χ_k be directional derivative in the directions $\mathbf{v}_k$.

Proposition 3.4.5. *We have*

$$\lambda = \alpha_{-1}\frac{h}{z} + \alpha_0 + \alpha_1\frac{z}{h} + \cdots .$$

We can find $\beta_{k,l}$ which are universal polynomials in the α_k so that

$$\sum_{l \le k} \beta_{k,l} \mathrm{Res}_{P(t)} \lambda^l \omega \equiv h^{2k-1}\mathbf{v}_{2k+1}(\omega) \mod h^{2k}.$$

Further,

$$\beta_{k,k} = n_k \alpha_{-1}^{n_k},$$

where $n_k \in \mathbf{Q}$ and $n_l \neq 0$.

Proof. First note that we can find $\gamma_{k,l}$, universal polynomials in the α_k, so that

$$\sum_{l \le k} \gamma_{k,l} \mathrm{Res}_{P(t)} \lambda^l \omega \equiv \alpha_{-1}^k \frac{h^k}{z^k}.$$

Indeed,

$$\lambda = \alpha_{-1}\frac{h}{z} + \cdots ,$$

$$\lambda^2 = \alpha_{-1}^2 \frac{h^2}{z^2} + 2\alpha_{-1}\alpha_0 \frac{h}{z} + \cdots ,$$

$$\lambda^3 = \alpha_{-1}^3 \frac{h^3}{z^3} + 3\alpha_0\alpha_{-1}^2 \frac{h^2}{z^2} + (3\alpha_{-1}\alpha_0^2 + 3\alpha_{-1}^2\alpha_1)\frac{h}{z} + \cdots .$$

So we can express

$$\mu_k = \alpha_{-1}^k \frac{h^k}{z^k}$$

in terms of $\lambda, \ldots, \lambda^k$ with the coefficients universal polynomials in the α_k's.

Note that $z-h$ vanishes on $R(t)$ and that $z-h$ is the anti-derivative of a holomorphic differential. Since the curve is hyperelliptic and $R(t)$ is a Weirstrass point, $z-h$ is an odd function under the involution of the hyperelliptic curve. However, any differential is even under the involution. Thus any differential can be expanded around $P(t)$ as a power series in the even powers of $z-h$ times dz. So if ω is a differential, we can write

$$\begin{aligned} \omega &= (a_0 + a_2(z-h)^2 + a_4(z-h)^4 + \cdots)dz \\ &= f((z-h)^2)dz. \end{aligned} \tag{25}$$

So

$$\mathrm{Res}_{P(t)}\mu_1\omega = \alpha_{-1}hf(h^2).$$

$$\mathrm{Res}_{P(t)}\mu_2\omega = 2h^3\alpha_{-1}^2 f'(h^2).$$

$$\mathrm{Res}_{P(t)}\mu_3\omega = h^3\alpha_{-1}^3(2f'(h^2) + 4h^2 f''(h^2)).$$

$$\mathrm{Res}_{P(t)}\mu_4\omega = h^4\alpha_{-1}^4(2hf''(h^2) + 8hf''(h^2) + 8h^3 f'''(h^2)).$$

Continuing in this way, we see that we can express

$$h^{2k-1}f^{(k)}(h^2)$$

as a linear combination of

$$\mathrm{Res}_{P(t)}\mu_l\omega$$

for $l \leq k$. On the other hand,

$$f^{(k)}(h^2) \equiv \mathbf{v}_{2k+1}(\omega) \mod h.$$

q.e.d.

Definition 3.4.6. Suppose $D_1, \ldots, D_n \in R_0[[\epsilon]]$. We assume $D_1, \ldots, D_n$ are linearly independent over $\mathbf{C}[[\epsilon]]$. Let $\mathcal{M}$ be the $\mathbf{C}[[\epsilon]]$ module generated by the D_k. Then we can find an increasing sequence of integers $m_1, \ldots, m_n$ and $E_1, \ldots, E_n \in R_0[[\epsilon]]$ so that the $E_k^{[0]}$ are all linearly independent over $\mathbf{C}$ and so that $\epsilon^{m_k} E_k$ are a basis of $\mathcal{M}$. $m_1, \ldots, m_n$ are uniquely determined and called the characteristic numbers of $D_1, \ldots, D_n$. The $\epsilon^{m_1} E_1, \ldots, \epsilon^{m_n} E_n$ are called a normalized basis for $\mathcal{M}$. Note that $E_1^{[0]}, \ldots, E_n^{[0]}$ are linearly independent over $\mathbf{C}$.

Corollary 3.4.7. *Suppose that T is one dimensional and $\{h = 0\}$ is just a point x and that h is a parameter near x. Let g be the function on $\mathbf{C}^n$ defined by*

$$g(L) = \mathbf{G}(x, L).$$

In the context of the Definition 3.4.6 suppose that $(\mathbf{G}, h; \chi, \chi_1, \ldots, \chi_n)$ is a representation of $D_1, \ldots, D_n$ and g does not satisfy any nonzero linear combination of $\mathbf{K}_1, \ldots, \mathbf{K}_n, E_1^{[0]}, \ldots, E_n^{[0]}$. Further assume that

$$\mathbf{v}_{2j+1}(g) \equiv \sum_{i \leq j} \beta_{i,j} \mathbf{K}_i(g) \mod h$$

with $\beta_{j,j} \neq 0$. Then the characteristic numbers of $D_1, \ldots, D_n$ are $1, 3, \ldots, 2n-1$. Further, the span of $E_1^{[0]}, \ldots, E_n^{[0]}$ is the same of the span of $\mathbf{K}_1, \ldots, \mathbf{K}_n$.

Proof. There are polynomials $P_{i,j} \in \mathbf{C}[\epsilon]$ so that

$$\sum_{i=1}^{j} P_{i,j}(h) \chi_i \equiv h^{2j-1} \mathbf{v}_{2j+1} \mod h^{2j}$$

with $P_{j,j}(0) \neq 0$. Then let

$$E_j' = \sum_{i=1}^{j} P_{i,j}(\epsilon) D_i.$$

Let $\epsilon^{r_j} F_j$ be the leading term of E_j' as a power series in ϵ. Then F_j is a linear combination of $E_1^{[0]}, \ldots, E_n^{[0]}$. If $r_j < 2j-1$, then

$$F_j(g) = 0.$$

So we would have to have $F_j = 0$. We conclude that $r_j \geq 2j-1$. On the other hand,

$$\mathbf{v}_{2j+1}(g) \equiv \sum_{i \leq j} \beta_{i,j} \mathbf{K}_i(g) \mod h$$

with $\beta_{j,j} \neq 0$. So $r_j \geq 2j-1$ and so $r_j = 2j-1$. Let

$$E_j = \epsilon^{-2j+1} E'_j.$$

Then

$$E_j^{[0]} = \sum_{i \leq j} \beta_{i,j} \mathbf{K}_i.$$

Since the $\mathbf{K}_i$ are independent, so are the $E_j^{[0]}$. q.e.d.

3.5

For future reference, we will now calculate the function $\mathbf{B}$ in a very special situation. Let T be a point and let $\mathcal{X}$ be a curve of arithmetic genus one with one node. We can find a normalization map $\pi : \mathbf{P}^1 \to \mathcal{X}$ so that $\pi(0) = \pi(\infty)$ is the node. Let p and s be points of $\mathbf{P}^1$, $q = 1/p$ and $r = 1$. We set $P = \pi(p)$, $Q = \pi(q)$, $S = \pi(s)$ and $R = \pi(r)$. Let δ_1 be the image of a circle traversed counterclockwise around $0 \in \mathbf{P}^1$. We let $\mathcal{M} = \mathcal{O}_{\mathcal{X}}(S)$. Then the function $\mathbf{B}$ will just depend on a number $\alpha \in \mathbf{C}$ and on p and s. The normalized differential is just

$$\omega_1 = \frac{1}{2\pi i} \frac{dz}{z}.$$

Explicitly, let

$$\mathcal{L}_\alpha = \mathcal{O}_{\mathcal{X}}(\pi(x) - S).$$

Then we should have

$$\frac{1}{2\pi i} \int_S^x \frac{dz}{z} = \alpha.$$

So

$$x = S e^{2\pi i \alpha}.$$

We use the usual parameter z on $\mathcal{X}$ coming from the parameter z on $\mathbf{P}^1$ to normalize our expressions for λ and the s_k. We can now write down these expressions using a degenerate theta function

$$\vartheta_k(z) = 1 - \frac{z}{k}.$$

So we get

$$\lambda(z) = \frac{\vartheta_1(z)^2 K_1}{\vartheta_p(z) \vartheta_q(z)},$$

where K_1 is chosen so that the first Laurent coefficient of λ at p is one. On the other hand,

$$\frac{s_{0,\alpha}}{s_{-1,\alpha}}(z) = \frac{\vartheta_p(z)\vartheta_{x/p^2}(z)K_2}{\vartheta_q(z)\vartheta_x(z)}$$

and define

$$r(z,t) = \frac{\vartheta_{t/p^2}(z)}{\vartheta_t(z)}.$$

Lemma 3.5.1.

$$\begin{aligned}\mathbf{B}_\alpha &= \frac{r(q,x)r(p,s)}{r(p,x)r(q,s)} \\ &= \frac{\vartheta_{x/p^2}(q)\vartheta_s(q)\vartheta_x(p)\vartheta_{s/p^2}(p)}{\vartheta_{x/p^2}(p)\vartheta_s(p)\vartheta_x(q)\vartheta_{s/p^2}(q)} \\ &= \frac{(1-p/x)(1-1/ps)(1-p/x)(1-p^3/s)}{(1-p^3/x)(1-p/s)(1-1/xp)(1-p/s)}.\end{aligned} \tag{26}$$

3.6

We will now consider how to use the family $\mathcal{X}_1 \to T_1$ to define representations. For each positive integer k, we define a map

$$\pi_k : \pi_*(\omega_{\mathcal{X}_1/T_1}) \to \mathcal{O}_{T_1}$$

by

$$\phi_k(\omega) = \mathrm{Res}_P(\lambda^k\omega),$$

where $\mathrm{Res}_{P(t)}(\lambda^k\omega)$ indicates the residue at $P(t)$ of the meromorphic section of the dualizing sheaf of $\mathcal{X}_{1,t} = C$ obtained by multiplying ω by λ^k. Notice that the ϕ_k all vanish on Z, since f_{-1} vanishes on Z. We define

$$\chi_k = \phi_k.$$

Lemma 3.6.1.

$$\psi = \frac{\chi_2 + 2\chi_1}{\epsilon}$$

vanishes on Z.

Proof. This follows from Proposition 3.4.1 (5). Indeed, using the fact that $\alpha_0 \equiv -1 \mod h$, we see that

$$\mathrm{Res}_{P(t)}(\lambda_{\mathrm{can}}^2 + 2\lambda_{\mathrm{can}})\omega$$

vanishes to order h^2. So $\chi_2 + 2\chi_1$ vanishes on Z. q.e.d.

We will initially assume C is smooth. We will also assume that

$$\int_Q^P v_k = hk.$$

We can now associate an infinite tridiagonal matrix C_p by defining its ij^{th} entry $C_{p,i,j}$ as

$$C_{p,i,j} = \begin{cases} 1, & \text{for } i = j+1 \\ A_{i,p} & \text{for } i = j \\ B_{i,p} & \text{for } i+1 = j \\ 0 & \text{otherwise.} \end{cases}$$

Note that C_p^n is a well-defined infinite matrix. If we have an infinite matrix E, we define a matrix E^+ by

$$E^+_{i,j} = \begin{cases} E_{i,j} & \text{for } i < j \\ 0 & \text{otherwise.} \end{cases}$$

We have identified $H^1(\mathcal{O}_C)$ with $\mathbf{C}^n$ using the I_k. So we may think of the ϕ_k as sections of the map $T_1 = T \times \mathbf{C}^n \to T$. So we can define vertical vector fields $\chi_1, \chi_2, \dots$ on T_1 corresponding to differentiating the direction $\phi_1, \phi_2, \dots$. We define χ to be differentiation in the direction $(1, 2, \dots, n)$.

Theorem 3.6.2.

$$\chi_k(C_p) = [(C_p^k)^+, C_p],$$

where $[E, F]$ *indicates the commutator of* E *and* F*.*

Proof. Assume that $h \neq 0$ and that $h = \frac{N}{M}$, with N and M integers. Then $M\gamma(Q) \in \mathbf{Z}^n$ and consequently there is a function α having a pole of order M at P and a zero of order M at Q. We normalize α so that α/λ^M has value one at P. Then we have

$$s_{k+M} = \alpha s_k$$

and consequently, A_k and B_k are periodic. Theorem 3.6.2 is just Theorem 4 of [7]. On the other hand, the set of points with h rational is dense and conclusion of this theorem holds generally. q.e.d.

We can find $T_k \in \mathcal{S}_1 \oplus \mathcal{S}_1$ so that $\mathcal{D}_{T_k}(A_l)$ is the l^{th} diagonal entry of $[(C_p^k)^+, C_p]$ and $\mathcal{D}_{T_k}(B_l)$ is the l^{th} off diagonal entry of $[(C_p^k)^+, C_p]$.

Definition 3.6.3. $D_k = D_{T_k}$.

For $t \in T$, let $\sigma : T \to T \times \mathbf{C}^n$ be defined by

$$\sigma(t) = (t, 1, 2, \ldots, n).$$

Then using Definition 3.4.2

$$\mathbf{A}_{k,x} = \mathbf{f}(x + kh(x)\sigma(\pi(x))$$

and

$$\mathbf{B}_{k,x} = \mathbf{g}(x + kh(x)\sigma(\pi(x)).$$

Theorem 3.6.4.

$$(\mathbf{f}, \mathbf{g}, \chi, \chi_1, \ldots, \chi_n)$$

form a representation of

$$D_1, \ldots, D_n.$$

Further, both $\mathbf{f}$ *and* $\mathbf{g}$ *are periodic under translation by* $(1, 2, \ldots, n)$. *Further,*

$$\mathbf{f}_1 = \frac{\mathbf{f} + 2}{h^2}$$

and

$$\mathbf{g}_1 = \frac{\mathbf{g} - 1}{h^2}$$

are meromorphic functions which are holomorphic at any point t *with* $h(t) = 0$ *provided that the* $\mathcal{N}_t$ *is non-special on the curve* $\mathcal{X}_{1,t}$. *Further, the* $D_1, \ldots, D_n$ *are* 1*-nice.*

Proof. We have checked the last statement when the curve $\mathcal{X}_{1,t}$ is smooth. But if $\mathcal{N}_t$ is non-special, then all the nearby line bundles on all the nearby curves are non-special. Hence $\mathbf{f}_1$ and $\mathbf{g}_1$ are analytic at points w near t provided that $h(w) \neq 0$. Further, at points w so that $h(w) = 0$, the two functions are holomorphic provided that $\mathcal{X}_{1,t}$ is smooth. Hence both functions are holomorphic on a neighborhood of t, except perhaps for a subset of codimension ≥ 2. So by Hartog's theorem, these functions are holomorphic at t. q.e.d.

Let $\psi = \chi_2 + 2\chi_1$. We can define P_1 and P_2 in S_2 so that

$$\mathcal{D}_{P_1,P_2}(A_k) = A_{k-1,p}B_{k-1,p} + A_{k,p}B_{k-1,p} + 2\,B_{k-1,p} \\ - A_{k,p}B_{k,p} - A_{k+1,p}B_{k,p} - 2\,B_{k,p},$$

and

$$\mathcal{D}_{P_1,P_2}(B_k) = B_{k,p}\Big(B_{k-1,p} + A_{k,p}^2 + 2\,A_{k,p} \\ - A_{k+1,p}^2 - 2\,A_{k+1,p} - B_{k+1,p}\Big).$$

Thus we have the tame derivation $D = D_{P_1,P_2}$ of $R[[\epsilon]]$.

Then we can use Theorem 3.6.2 to calculate:

$$\psi(A_{k,p}) = \mathcal{D}_{P_1,P_2}(A_k),$$

$$\psi(B_{k,p}) = \mathcal{D}_{P_1,P_2}(B_k).$$

Proposition 3.6.5. $(\mathbf{f}, \mathbf{g}, h, \chi, \psi)$ *form a representation of* D_{P_1,P_2}. *Further,* ψ *is slow for this representation.*

Let's calculate the first terms of D_Φ as a series in ϵ. We get

$$D(v^{(0)}) = E_{-1}(v^{(0)})E_{-1}(w^{(0)}) + (v^{(0)})E_{-1}(w^{(0)}) \\ + 2E_{-1}(w^{(0)}) - (v^{(0)})w^{(0)} - E_1(v^{(0)})w^{(0)} - 2w^{(0)}. \tag{27}$$

We have

$$D_\Phi(v^{(0)}) = \Phi(D(v^{(0)}).$$

So

$$D_\Phi(\epsilon^2 v^{(0)}) = (E_{-1}(-2 + \epsilon^2 v^{(0)})E_{-1}(1 + \epsilon^2 w^{(0)}) \\ + (-2 + \epsilon^2 v^{(0)})E_{-1}(1 + \epsilon^2 w^{(0)}) \\ + 2E_{-1}(1 + \epsilon^2 w^{(0)}) - (-2 + \epsilon^2 v^{(0)})(1 + \epsilon^2 w^{(0)}) \\ - E_1(-2 + \epsilon^2 v^{(0)})(1 + \epsilon^2 w^{(0)}) - 2(1 + \epsilon^2 w^{(0)}). \tag{28}$$

Evaluating we get

$$D_\Phi(v^{(0)}) = \epsilon(v^{(1)} - w^{(1)}) + \text{higher order terms in } \epsilon.$$

Similarly,

$$D_\Phi(w^{(0)}) = \epsilon(-v^{(1)} + w^{(1)}) + \text{higher order terms in} \epsilon.$$

Definition 3.6.6. $\mathcal{D}_k = D_{k,\Phi}$.

Theorem 3.6.7.

$$(\mathbf{f}_1, \mathbf{g}_1, \chi, \chi_1, \ldots, \chi_n)$$

form a representation of

$$\mathcal{D}_1, \ldots, \mathcal{D}_n.$$

Further, both $\mathbf{f}_1$ *and* $\mathbf{g}_1$ *are periodic under translation by* $(1, 2, \ldots, n)$. *Further, set*

$$\mathcal{D}_1' = \frac{\mathcal{D}_2 + 2\mathcal{D}_1}{\epsilon}$$

and

$$\chi_1' = \frac{\chi_2 + 2\chi_1}{h}.$$

Then the representation

$$(\mathbf{f}_1, \mathbf{g}_1, \chi, \chi_1')$$

is slow for $\mathcal{D}_1'$.

3.7

We will be applying this construction of slow representations to prove Theorem 1.0.1. We need to have a criterion for checking that such representations are general.

Definition 3.7.1. Let S be a complex manifold and let $\pi : X \to S$ be a family of stable curves parameterized by S. Let $s_0 \in S$ be a point and suppose that $N_1, N_2, \ldots, N_p$ are the nodes of X_{x_0}. Then we can find functions f_k defined near s_0 so that the deformation of the node N_k is locally isomorphic to $xy = f_k$. We say the nodes are independent at s_0 if the differentials of the f_k are independent at x_0.

Remark 3.7.2. Suppose the $g_1, \ldots, g_r$ are functions on S. Suppose that the $g_1, \ldots, g_r$ have independent differentials when restricted to some smooth S' on which all the f_k from the above definition vanish. Let S'' be submanifold defined by the vanishing of all the g_i. Then the family $X \times_S S''$ has independent nodes.

We will consider the following situation: Let $\mathcal{M}$ be a connected complex manifold and let Λ be a sheaf of abelian groups locally isomorphic to $\mathbf{Z}^{2n}$. We will assume that Λ has a symplectic form $\langle\ ,\ \rangle$. Let p be a point of $\mathcal{M}$. There is the usual monodromy representation ρ of $\pi_1(\mathcal{M}, p)$

on the stalk Λ_p. Suppose that there exist $\delta_1, \delta_2, \dots, \delta_n$ in Λ_p so that the endomorphisms T_i of Λ_p defined by

$$T_i(\gamma) = \gamma + \langle \gamma, \delta_i \rangle \delta_i$$

are all in the image of ρ. We also assume that $\langle \delta_i, \delta_j \rangle = 0$ for all i and j. Let

$$\delta = \delta_1 + 2\delta_2 + \cdots + n\delta_n.$$

Let $\pi : V \to \mathcal{M}$ be a vector bundle of rank n. Here V is a physical bundle, i.e., V is a complex manifold and for $q \in \mathcal{M}$, the fibers of π, $\pi^{-1}(q) = V_q$, are given the structure of complex vector spaces. Let $\mathcal{V}$ be the sheaf of analytic sections of V, so that $\mathcal{V}$ is locally free of rank n on $\mathcal{M}$. Suppose that Λ is a subsheaf of $\mathcal{V}$ so that each $\lambda \in \Lambda_p$ can be considered a local section of V defined around p. In particular, by evaluating at p, we get a map μ_p from $\Lambda_p \to V_p$. We assume that the image of μ_p is a lattice in V_p. We assume that the images of the δ_i give a complex basis of V_p for each p.

Let f be a meromorphic function on V. In particular, we can look at f_p, the restriction of f to V_p. We will assume that f_p is invariant under translation by elements of Λ_p for all p.

Definition 3.7.3. Let W be a complex subbundle of V. We say that W is good if f is constant on all the cosets of $W_p \subset V_p$ and the elements of $\Lambda_p \cap W_p$ span W as a complex vector space.

Let $U \subset \mathcal{M}$ be an connected open set and let λ be a section of Λ over U. The set of $p \in U$ so that $\lambda(p) \in W_p$ is either all of U or is defined by nontrivial analytic conditions and so is nowhere dense. Consequently, we may find a set of the second category $T \subset \mathcal{M}$ so that if $p \in T$ and $\lambda(p) \in W_p$, then $\lambda(q) \in W_q$ for all $q \in U$. We call such a point very general.

If W is good, then the sheaf $\Lambda_W = \Lambda \cap W \subset \Lambda$ is a locally isomorphic to $\mathbf{Z}^k$ for some k and $\Lambda_W \otimes \mathcal{O}_{\mathcal{M}} = W$.

Lemma 3.7.4. *Suppose that the subbundle of V generated by δ is good. Then f_p is constant.*

Proof. It suffices to prove the assertion for a very general point p. Let W be a good subbundle. First note that the monodromy representation ρ leaves $L_p = \Lambda_p \cap W_p$ invariant, since p is very general.

Let Q_p be a maximal complex subspace of V_p so that f_p is constant on the cosets of Q_p. Note that if f_p is constant on the cosets of Q_1

and Q_2, then there f_p is constant on the cosets of $Q_1 + Q_2$, so such a maximal Q_p exists. First, suppose that $L = Q_p \cap \Lambda_p$ is not a lattice in Q_p. Let $q : V_p \to V_p/\Lambda_p$ be the quotient map and consider the closure X of $q(Q_p)$ in the torus V_p/Λ_p. Note that f can be considered a function on V_p/Λ_p which is constant on the cosets of $q(Q_p)$ and hence on the cosets of X. But X is a closed subgroup of V_p/Λ_p and hence there is a real subspace Q'_p of V_p so that $q(Q'_p) = X$ and f_p is constant on the cosets of Q'_p. Further, $Q'_p \cap \Lambda_p$ is a lattice in Q'_p. f_p is meromorphic, so f_p is constant on the complex subspace Q''_p spanned by the vectors in Q'_p. Thus $Q''_p = Q_p$, so $L_p = Q_p \cap \Lambda_p$ is a lattice in Q_p.

For any point p, there is a simply connected neighborhood $U \subset \mathcal{M}$ of p and a set $T \subset U$ of the second category in U so that $Q_q \cap \Lambda_q = Q_r \cap \Lambda_r$ for all q and r in T, since the set of subgroups of $\mathbf{Z}^{2n}$ is denumerable. Here we have identified Λ_r with Λ_q, since they are both identified with the global sections of Λ over U. We can then find a subsheaf $\mathcal{W}$ of $\mathcal{V}$ so that $\mathcal{W}_q = Q_q$ for all $q \in T$. Our sheaf $\mathcal{W}$ has been constructed in a neighborhood of a arbitrary point p, but these sheaves constructed at different points coincide on the overlaps, since their fibers coincide over sets of the second category. Consequently, we have a sheaf $\mathcal{W}$ so that the fiber of $\mathcal{W}$ at q is Q_q for a dense set of q. Let $\Lambda_W = \mathcal{W} \cap \Lambda$. Let $W \subset V$ be the physical bundle associated with $\mathcal{W}$. Then f is constant on the cosets of W_q for a dense set of q. Consequently, $\mathcal{W}_p \subseteq Q_p$ for all p with equality for a set of the second category T.

Assume that $p \in T$. Then monodromy operates on $\mathcal{W}_p = Q_p$. Let U_p be the real span of the δ_i . The δ_i form a complex basis of V_p, so the real dimension of $Q_q \cap U_p$ is less than or equal to the complex dimension of Q_p. Also note that we can find a symplectic form $\langle\ ,\ \rangle$ on V_p as a real vector space extending the form on Λ. Observe that if $v \in Q_p$ and $\langle v, \delta_k\rangle \neq 0$, then $\delta_k \in Q_p$, since monodromy acts. Consider the map $T : Q_p \to Q_p \cap U_p$ defined by

$$T(v) = \sum_k \langle v, \delta_k\rangle \delta_k.$$

The observation shows that T does map to U_p. Since U_p is maximal isotropic, the kernel of T is contained in $Q_p \cap U_p$. So the real dimension of Q_p is less than or equal to twice the real dimension of $Q_p \cap U_p$ and the map T is onto. The dimension of the image of T is the number of k so that $\langle v, \delta_k\rangle \neq 0$ for some $v \in Q_p$. Hence $Q_p \cap U_p$ has a real basis consisting of some subset of the δ_k. But $\delta \in Q_p \cap U_p$ and δ is linear combination of the δ_k so that all the δ_k appear nontrivially in δ. So $Q_p = V_p$ and f_p is constant. q.e.d.

4. Explicit construction of curves

4.1

Our aim is to construct a family of representations which are generic and general and then to apply Theorem 2.9.5. Theorem 3.6.7 already checks that our families will satisfy conditions (2), (3) and (4) of Theorem 2.9.5, except that we want to use the following for the definition of D_1 in Theorem 2.9.5:

$$D_1 = \frac{\mathcal{D}_2 + 2\mathcal{D}_1}{\epsilon}.$$

To construct the n^{th} representations, we will be considering families of curves in $\mathbf{P}^1 \times \mathbf{P}^1$ over $\mathbf{C}$, which are generically double sheeted coverings of the second factor. We will suppress the n in the notation until we are ready to put all the representations into a family. Let

$$T_0 = H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(n+1, 2)).$$

So an element of T_0 is a polynomial in the variable X_0, X_1, Y_0, Y_1, which is homogeneous of degree $n+1$ in X_0, X_1 and homogeneous of degree 2 in Y_0, Y_1. Usually, we will use affine coordinates to describe the elements of T_0 we will be considering , where we set $X_0 = 1$, $X_1 = x$, $Y_0 = 1$ and $Y_1 = y$. One can easily pass from the affine coordinates to the homogenous coordinates. So the elements of T_0 of interest to us can be described by

$$a(x)y^2 + b(x)y + c(x).$$

Let $T_1 \subset \mathbf{P}^1 \times \mathbf{P}^1 \times T_0$ be the universal curve and let $\pi_3 : T_1 \to T_0$ and $\pi_1 : T_1 \to \mathbf{P}^1$ be the obvious projections.

There is an birational involution ι of T_1 over T_0 given by

$$\iota(x, y) = \left(x, -\frac{b(x)}{a(x)} - y\right).$$

Note that ι is only defined for those points (x, y) with $a(x) \neq 0$.

Consider the map Λ from $\mathbf{C}^n$ to T_0:

$$\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_n) = (y^2 - x)(x - \alpha_1)\ldots(x - \alpha_n) = P_{\alpha_1,\ldots,\alpha_n}.$$

Let L_0 be the image of Λ in T. Let

$$C(\alpha_1, \alpha_2, \ldots, \alpha_n)$$

be the curve

$$\pi_0^{-1}(\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_n)).$$

The α_k form a partial system of local coordinates around any point $\Lambda(\alpha_1, \alpha_2, \ldots, \alpha_n)$ on T_0 as long as the α_k are distinct and so L_0 is a submanifold of T_0 at those points.

Let P_0 denote $\Lambda\left(1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}\right) \in T_0$. We will be investigating curves in a neighborhood of

$$C_0 = C\left(1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots, \frac{1}{n^2}\right).$$

For P near P_0, $T_1 \to T_0$ forms a family of semi-stable curves parameterized by a neighborhood $U \subset T_0$ of P_0. On C_0, we have $2n$ nodes $N_k = (k^2, k)$ for k between $-n$ and -1 and between 1 and n.

Lemma 4.1.1. *The nodes of the family $T_1 \to T_0$ are independent near P_0. (See Definition 3.7.4 for the meaning of independent.)*

Proof. By replacing U by a smaller neighborhood of P_0, we can find f_k defined on U so that the deformation of the node N_k is locally given by

$$\left(x - \frac{1}{k^2}\right)(y^2 - x) + f_k(P)$$

for $P \in T_0$ near P_0. The f_k have independent differentials at P_0. Indeed, it suffices to show that for any k, we can construct a map $\psi_k : D \to T_0$, where D is the unit disk, so that $\psi_k^*(f_p)$ vanishes identically if $p \neq k$, but vanishes to exactly order one at $0 \in D$ if $p = k$.

$$W(x, y, t)$$
$$= \left((y^2 - x)\left(x - \frac{1}{k^2}\right) + \left(t^2 + \frac{2t}{k}\right)\left(y + \frac{1}{k}\right)^2\right) \prod_{p^2 \neq k^2} \left(x - \frac{1}{p^2}\right).$$

Note that

$$\frac{\partial W}{\partial t}\left(\frac{1}{k^2}, \frac{1}{k}, 0\right) \neq 0$$

so that the total space family of curves over D defined by $W = 0$ is smooth at $(\frac{1}{k^2}, \frac{1}{k}, 0)$, so that $\psi_k^*(f_k)$ vanishes exactly once at $t = 0$. On the other hand, $(\frac{1}{k^2}, -\frac{1}{k})$ continues to be a node of the curve $W(x,y,t) = 0$ so $\psi^*_{-k}(f_k) = 0$ and two distinct nodes continue to lie over $x = p^2$ for $p^2 \neq k^2$. So $\psi^*_p(f_k) = 0$. q.e.d.

4.2

For $k > 0$, choose small circles β_k oriented counterclockwise around the points $\frac{1}{k^2}$ and let $\delta_{k,0}$ be the lift of β_k to C_0 which is near to the point $(\frac{1}{k^2}, \frac{1}{k})$ so that $\pi_1 \circ \delta_{k,0} = \beta_k$. For some neighborhood $U \subset T_0$ of P_0, we can find a map $\delta_k : S^1 \times U \to \pi_3^{-1}(U)$ defined over U which restricts to δ_{k,P_0}, where $\pi_3 : T_1 \to T_0$ is the projection. Let ω_{T_1/T_0} be the sheaf of relative dualizing differentials. Let $\delta_{k,Q}$ be the cycle $\delta_k(S^1, Q)$ on $T_{1,Q}$ for $Q \in T_0$. By possibly shrinking U, a section w of ω_{T_1/T_0} over $V \subset U$ can be integrated fiberwise over the cycle $\delta_{k,Q}$. Thus we obtain a maps over V,

$$\int_{\delta_k} : \pi_{3,*}(\omega_{T_1/T_0}) \to \mathcal{O}_{T_0},$$

i.e.,

$$\left(\int_{\delta_k} \omega\right)_Q = \int_{\delta_k(S^1,Q)} \omega_Q.$$

Thus we get a map

$$\Psi : \pi_{3,*}(\omega_{T_1/T_0}) \to \bigoplus_k \mathcal{O}_{T_0}$$

as the direct sum of the $\int_{\delta_k}$.

Note that we can compute the dualizing differentials on $D = C(\alpha_1, \ldots, \alpha_n)$. Namely, let w_k be the differential

$$\frac{\sqrt{\alpha}_k dy}{\pi i(y^2 - \alpha_k)}.$$

Then w_k extends to a section of ω_D and

$$\int_{\delta_p} w_k = \delta_{p,k},$$

where $\delta_{p,k}$ indicates the Kronecker delta function. So by shrinking U again to a neighborhood of P_0, we can assume that Ψ is an isomorphism. We have established:

Lemma 4.2.1. *We can find local sections v_p of $\pi_{3,*}(\omega_{T_1/T_0})$ so that*

$$\int_{\delta_k} v_p = \delta_{k,p}.$$

A point $t = (E, p)$ of T_1 consists of an equation $E(x, y) \in T_0$ for a curve $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ and a point $p \in C$. Let U_1 be a small neighborhood of $(P_0, (0, 0))$. We introduce functions h_p of $z \in U_1$ by the formula

$$h_p(z) = \int_{\iota(z)}^{z} v_p,$$

where we define this when the projection of z to $\mathbf{P}^1 \times \mathbf{P}^1$ near zero and $P = \pi_3(z)$ is near P_0. Here we must specify a path γ from $\iota(z)$ to y. First we ask that $\pi_3(\gamma)$ be a point and that the projection of γ to $\mathbf{P}^1 \times \mathbf{P}^1$ lie near $(0, 0)$. Notice that on C_0, the point $(0, 0)$ is fixed under ι. So as long as we stay near to $(0, 0)$ and C_0, it makes sense to ask that the path from stays near $(0, 0)$. With these assumptions, h_p is well-defined. The functions h_k all vanish on the ramification locus $\mathcal{R}$ of the map $\pi_1 \times \pi_3 : U \to \mathbf{P}^1 \times T_0$, when the h_k are defined, for $\mathcal{R}$ is just defined by $\iota(z) = y$. Also $\pi_3 : \mathcal{R} \to T_0$ is a local isomorphism at the points we are considering.

4.3

We can compute these h_k on the curves $C(\alpha_1, \ldots, \alpha_n)$. The projection of our path γ lies on the curve $C(\alpha_1, \ldots, \alpha_n)$. Let the projection of z to the second factor of $\mathbf{P}^1 \times \mathbf{P}^1$ be y_0.

$$\begin{aligned} h_k(z) &= \int_{-y_0}^{y_0} \frac{\sqrt{\alpha}_k dy}{\pi i(y^2 - \alpha_k)} \\ &= \frac{\sqrt{\alpha}_k}{\pi i} \log\left(\frac{\alpha_k - y_0}{\alpha_k + y_0}\right) \\ &\approx \frac{2y_0}{\pi i \sqrt{\alpha}_k}. \end{aligned} \tag{29}$$

where $\approx$ indicates approximately when y_0 is close to zero.

This means that each of the $h_k = 0$ defines $\mathcal{R} \cap \pi_3^{-1}(L_0)$ as a subscheme of $\pi_3^{-1}(L_0)$ in a neighborhood of P_0, since $y = 0$ vanishes to order one $\mathcal{R} \cap \pi_3^{-1}(L_0) \subset \pi_3^{-1}(L_0)$. Since all the h_k vanish on $\mathcal{R}$ in a neighborhood of $(0, 0, y)$, we see that all the h_k vanish to order one along $\mathcal{R} \cap \pi_3^{-1}(L_0)$.

Let

$$H_k = \frac{h_k}{h_1}$$

for k from 2 to n. $\mathcal{R} \cap \pi_3^{-1}(L_0)$ can be identified with L_0 locally via π_3, so we can use the coordinates $\alpha_1, \ldots, \alpha_n$ as coordinates on $\mathcal{R} \cap \pi_3^{-1}(L_0)$. Then when restricted to $\mathcal{R} \cap \pi_3^{-1}(L_0)$, the $(H_k)_{|\mathcal{R} \cap \pi_3^{-1}(L_0)}$ just become

$$(H_k)_{|\mathcal{R} \cap \pi_3^{-1}(L_0)} = \sqrt{\frac{\alpha_k}{\alpha_1}}.$$

Note that the equations $h_1, H_2, \ldots, H_n$ all have independent differentials at $(0, 0, P_0)$, since the H_k have independent differentials when restricted to $\mathcal{R} \cap \pi_3^{-1}(L_0)$.

Lemma 4.3.1. *We have*

$$\frac{\partial}{\partial \alpha_k} H_k \neq 0$$

near $(0, 0), P_0$.

4.4

Let

$$T = \{t \in T_1 | H_k(t) = k\}$$

and let

$$\mathcal{X} = T \times_{T_0} T_1.$$

The map $\pi : \mathcal{X} \to T$ has a canonical section $P : T \to \mathcal{X}$ defined in the following way: A point t of T consists of an equation $E(x, y) \in T_0$ for a curve $C \subset \mathbf{P}^1 \times \mathbf{P}^1$ and a point $p \in C$. $\pi^{-1}(t)$ is canonically identified with C. So we define $P(t) = p$. We have another section $Q : T \to \mathcal{X}$ defined by $Q(t) = \iota(p)$. On the other hand, the ramification locus $\mathcal{R} \subset U$ is locally isomorphic to T_0. So we can find a section $R : U' \to \mathcal{X}$ so that $\iota(R(t)) = R(t)$, where U' is a neighborhood of $(0, 0, P_0)$. By shrinking U, we can assume that the images of $P(t)$, $Q(t)$ and $R(t)$ in $\mathbf{P}^1 \times \mathbf{P}^1$ are all near $(0, 0)$. By localizing on T, we can assume that there is a section s of $\mathcal{O}(2R(t))$ which is not constant when restricted to any fiber of π. By possibly further restricting T we can a map $\mathbf{z}$ of a neighborhood of $\mathcal{R}$ to $T \times D$ over T so that

$$s = \frac{1}{\mathbf{z}^2}.$$

Using Lemma 3.7.4, we see:

Lemma 4.4.1. *T is smooth near $(P_0,(0,0)) = t_0$. $\pi : \mathcal{X} \to T$ is a family of semi-stable curves . $\mathcal{X}_{t_0}$ has $2n$ nodes and these nodes are independent. The I_{δ_k} for $k > 0$ form a basis of the $R^1\pi_*(\mathcal{O})$ locally. Let T_2 be the subset of T defined by $h = 0$ and let $X_1 = \pi^{-1}(T_2)$. Let V be the physical bundle associated to $R^1\pi_*(\mathcal{O}_{X_1})$ Let f be a meromorphic function V defined on a neighborhood of the inverse image of t_0. Then if f is constant on the fibers of the line bundle generated by*

$$\delta = \delta_1 + \delta_2 + \cdots + \delta_n,$$

then f is constant on the fibers of V.

At this point, we will make a choice of a line bundle $\mathcal{M}$ on T_1. Note that

$$T_1 \subset T_0 \times \mathbf{P}^1 \times \mathbf{P}^1.$$

A point of T_1 consists of an equation $E \in T_0$ and a point (x, y) with $E(x, y) = 0$. So we can map $\phi : T_1 \to \mathbf{P}^1$ by $\phi(E, x, y) = y$. On the other hand, locally around P_0, we can find a map $\gamma : U_0 \to T_1$ so that $\gamma(U_0) \subset \mathcal{R}$. Thus $\gamma(U_0)$ is a divisor on the inverse image U_1 of U in T_1. We let

$$\mathcal{M} = \phi^*(\mathcal{O}_{\mathbf{P}^1}(1)) \otimes \mathcal{O}_{U_1}(-\gamma(U_0)).$$

We denote the pullback of $\mathcal{M}$ to $\mathcal{X}$ by $\mathcal{M}$ again. We now have a function $\mathbf{B}(z, \alpha_1, \alpha_2, \ldots, \alpha_n)$ defined locally on $T_1 - T_2$. Thus $\mathcal{M}$ restricted to the curve $\{E = 0\} = C$ is just

$$\phi^*(\mathcal{O}_{\mathbf{P}^1}(1)) \otimes \mathcal{O}_C(-R)$$

on the curve $C(\alpha_1, \alpha_2, \ldots, \alpha_n)$. This bundle has degree one on all the vertical components of $C(\alpha_1, \alpha_2, \ldots, \alpha_n)$, but degree zero on the curve $\{y^2 = x\}$.

4.5

Let b be a nonzero integer between $-n$ and n. Let

$$W_t(x, y) = ((y^2 - x)(x - b^2) + (t^2 - 2bt)(y - b)^2)).$$

Note that the point (b^2, b) is a node the curve $W_t = 0$ for all t. Fixing t, let (x_1, y_1) be a generic point of $W_t = 0$. Let

$$\mathcal{S} = \prod_{k \neq b}(x - g_k).$$

Let $\mathcal{Z} = (W_t\mathcal{S}, (x_1, y_1)) \in T_1$. Our aim is to evaluate

$$\mathbf{B}(\mathcal{Z}, \alpha_1, \alpha_2, \ldots, \alpha_n).$$

We can find a map of $\psi_{1,t} : \mathbf{P}^1 \to \mathbf{P}^1$ by

$$\psi_{1,t}(w) = -\frac{4\,wt^2 - 8\,wt\,b - b^2 + 2\,b^2w - b^2w^2}{(w+1)^2}$$

and $\psi_{2,t} : \mathbf{P}^1 \to \mathbf{P}^1$ by

$$\psi_{2,t}(w) = -\frac{4\,wt^2 - 8\,wt\,b + bt - 2\,b^2 - bw^2t + 2\,b^2w}{(wt - t + 2\,b)\,(w+1)}.$$

Let $\psi_t = (\psi_{1,t}, \psi_{2,t}) : \mathbf{P}^1 \to \mathbf{P}^1 \times \mathbf{P}^1$. Then ψ_t maps $\mathbf{P}^1$ to the curve $W_t = 0$ and is in fact the normalization of this curve for t generic. Further, we have $\psi_{1,t}(0) = \psi_{1,t}(\infty) = b^2$ and $\psi_{2,t}(0) = \psi_{2,t}(\infty) = b$, so 0 and ∞ map to the node of $W_t = 0$. Further, $\psi_{1,t}$ is ramified at 1 and in fact $\psi_{1,t}(w) = \psi_{1,t}(1/w)$. Now $\psi_{2,t}^{-1}(\infty) = \{-1, 1 - 2b/t\} = \{P_1, P_2\}$. Let $C \subset \mathcal{X}_p$ be the curve with equation $W_t = 0$.

Let $\mathcal{L}$ be any line bundle on $\mathcal{X}_z$ which has degree zero on all the irreducible components of $\mathcal{X}_z$.

Lemma 4.5.1. *The natural restriction map $\phi : H^0(\mathcal{X}_z, \mathcal{L} \otimes \mathcal{M}_z) \to H^0(C, \mathcal{L} \otimes \mathcal{M}_z \otimes \mathcal{O}_C)$ is an isomorphism.*

Proof. Note that $\mathcal{L} \otimes \mathcal{M}_z \otimes \mathcal{O}_C$ has degree one on a curve of arithmetic genus one, while $\mathcal{L} \otimes \mathcal{M}_z$ has degree n on a curve of arithmetic genus n. Hence, we need only show that ϕ is injective. A section s in the kernel of ϕ is a section of $\mathcal{L} \otimes \mathcal{M}_z$ which vanishes on the curve C. The other components of $\mathcal{X}_z$ are all fibers of the projection of $\mathbf{P}^1 \times \mathbf{P}^1$ onto the first factor. As such, the degree of $\mathcal{M}_z$ on these components is one. But these components meet C in two points. So the restriction of s to these components is a section of a bundle of degree one which vanishes at two points. Hence the section vanishes on all the vertical components, and so $s = 0$. q.e.d.

For $v_0 \in \mathbf{P}^1$, let $\psi_t(v_0) = (x_1, y_1)$. We will develop conditions on the g_k so that $\mathcal{Z} \in T$. In fact, we will write g_k as a function of v_0 and t. We can write

$$w_k = \frac{1}{2\pi i}\left(\frac{1}{z - g_k} - \frac{1}{z - 1/g_k}\right) dz$$

for $k \neq 0$, while

$$w_0 = \frac{1}{2\pi i}\frac{dz}{z}.$$

We will consider g_k close to
$$\frac{b-k}{b+k}.$$
The w_k are the pullbacks of the canonical differentials on curve in $\mathbf{P}^1 \times \mathbf{P}^1$ corresponding to $\mathcal{Z}$. Then we have

$$\begin{aligned}
h_k &= \int_{1/v_0}^{v_0} w_k \\
&= \frac{1}{2\pi i}\Big(\log(v_0 - g_k) - \log(1/v_0 - g_k) \\
&\quad - \log(v_0 - 1/g_k) + \log(1/v_0 - 1/g_k)\Big) \\
&= \frac{1}{\pi i}\log\left(\frac{v_0 - g_k}{1 - g_k v_0}\right).
\end{aligned} \tag{30}$$

Note that we have chosen the usual branch of the log so that h_k vanishes when $v_0 = 1$. We have

$$\begin{aligned}
h_b &= \int_{1/v_0}^{v_0} w_b \\
&= \frac{1}{\pi i}\log(v_0).
\end{aligned} \tag{31}$$

Next we choose $g_1, \dots, g_{b-1}, g_{b+1}, \dots$ so that
$$H_k = k.$$

We do this by first choosing g_1 as a function of v_0 and t to make $h_b = b h_1$. Indeed, we can just take
$$g_1 = v_0 \frac{1 - v_0^{b-1}}{1 - v_0^{b+1}}.$$

Note that g_1 is analytic even when $v_0 = 1$ and in fact
$$g_1 = \frac{b-1}{b+1},$$
when $v_0 = 1$. We can find similar formulas for g_k in terms of v_0 for the rest of the k which are not b. Let
$$R(t, v_0)(x, y) = W_t(x, y) \prod_{j \neq b} (x - \psi_{1,t}(g_k(v_0, t))).$$

Recall that

$$h = \frac{1}{b} h_b = \frac{1}{bi\pi} \log(v_0)$$

so that $v_0 = \exp(bi\pi h)$. Then we can construct a map

$$\Phi_b : \mathbf{C} \times \mathbf{C} \to T$$

by

$$\Phi_b(h,t) = (R(t, v_0), (x_1, y_1)).$$

Note that $\Phi_b(0,0) = (P_0, (0,0))$. Then $\Phi_b(h,t) \in T$. We will use Lemma 3.5.1 to compute

$$\mathbf{B}(\Phi_b(h,t), \alpha_1, \ldots, \alpha_n).$$

Let $s = 2b/t - 1$. Note that the pullback of $\mathcal{M}$ to curve $C = \{W_t = 0\}$ is $\mathcal{O}_C(\psi_t(S))$, where $S = \pi(s)$. Let $x = se^{2\pi i \alpha_b}$. Then

$$\mathbf{B}(\Phi_b(h,t), \alpha_1, \ldots, \alpha_n) = \frac{(-x + v_0)^2 (v_0 s - 1) \left(-s + {v_0}^3\right)}{(v_0 x - 1) (-s + v_0)^2 \left(-x + {v_0}^3\right)}.$$

Let

$$\mathbf{H} = \left(\frac{-1 + \mathbf{B}(\Phi_b(h,t), \alpha_1, \ldots, \alpha_n)}{h^2} \right)_{h=0}.$$

Then we can compute

$$\mathbf{H} = 4 \, \frac{s \left(s^2 \beta^2 - \beta + 1 - s^2 \beta\right) \pi^2}{(s\beta - 1)^2 (s - 1)^2},$$

where

$$\beta = \frac{x}{s} = \exp(2i\pi\alpha_b).$$

So we get:

Lemma 4.5.2.

$$\left(\frac{d\mathbf{H}}{dt} \right)_{t=0} = \frac{2\pi^2(\beta - 1)b}{\beta}.$$

4.6

Recall that the family $\pi : \mathcal{X} \to T$ depends on the integer n. (The curves have bidegree $(2, n+1)$.) Let's rename T as V_n and Φ_b as $\Phi_{b,n}$ and $\Phi_b(0,0) = Q_n$. We also define

$$\phi_{b,n}(t) = \Phi_{b,n}(0,t).$$

We also denote $N_n = V_n \times \mathbf{C}^n$. We claim that we can fit the N_n into a compatible family. Our first task is to construct a map $k_n : V_n \to V_{n+1}$. Let

$$T_{0,n} = H^0(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(n+1,2)).$$

We map $u_n : T_{0,n} \times \mathbf{C} \to T_{0,n+1}$ by

$$u_n(\mathcal{P}, \beta)(x,y) = \mathcal{P}(x,y)(x-\beta).$$

Here we will only deal with β near $1/(n+1)^2$ so the curve $x - \beta = 0$ will meet the curve $P(x,y) = 0$ transversally. Now we see that the fiber $C_n(\mathcal{P})$ of $T_{1,n} \to T_{0,n}$ over $\mathcal{P}$ is naturally a subcurve of the fiber of $C_{n+1}(u_n(\mathcal{P}))$, where $T_{1,n}$ is the universal curve over $T_{0,n}$. Thus we have natural maps

$$k_n : T_{1,n} \to T_{1,n+1}.$$

Further, the normalized differentials $v_{k,n+1}$ on the curve $C_{n+1}(\mathcal{P})$ restrict to the normalized differentials $v_{k,n}$ for $k = 1, \ldots, n$. So the functions

$$h_{k,n}(\mathcal{P}, \beta) = \int_Q^P v_{k,n+1}$$

on the curve $C_{k_n(\mathcal{P},\beta)}$ are independent of β and in fact

$$h_{k,n}(\mathcal{P}, \beta) = \int_Q^P v_{k,n},$$

where the latter integral is taken on the curve $\mathcal{P}(x,y) = 0$. On the other hand,

$$\frac{\partial}{\partial \beta} \frac{h_{n+1,n+1}}{h_{1,n+1}} \neq 0$$

when β is near

$$\frac{1}{(n+1)^2}$$

by Lemma 4.5.2.

Now

$$V_n \subset T_{1,n}$$

is the set of $(\mathcal{P}, P)$ with $P \in \mathcal{C}_n(\mathcal{P})$ so that

$$\int_{\iota(P)}^{P} v_{k,n} = k \int_{\iota(P)}^{P} v_{1,n} = kh_{1,n}.$$

The functions k_n map $V_n \to V_{n+1}$. Hence we have functions $\mathbf{f}_n$ and $\mathbf{g}_n$ on $V_n \times \mathbf{C}^n$ which form a representation of $D_1, \ldots, D_n, \ldots$. Further,

$$\mathbf{f}_n(v, z_1, z_1, \ldots, z_n) = \mathbf{f}_{n+1}(k_n(v), z_1, \ldots, z_{n+1}).$$

We let V_n', $\mathbf{f}_n'$ be the extended representations. (Definition 2.6.5.)

Let g_n' be the meromorphic functions defined on $(V_n' \times \mathbf{C}) \cap \{h_{1,n}\} = 0$

$$g_n'(v, z) = \mathbf{g}_{1,n}'(v, z, 2z, \ldots, nz).$$

Lemma 4.6.1. *The family $\{g_n'\}$ is generic.*

Proof. Let b be an nonzero integer between $-n$ and n. Then Lemma 4.5.2 shows that there are maps $\phi_{b,n}$ from the disk $D \times \mathbf{C}$ to V_n so that

$$\left(\frac{\partial \phi_{b,n}^*(g_{1,n})}{\partial t}\right)(0, z) = \frac{4\pi^2}{b}(1 - \exp(-2\pi i b z)).$$

We can then consider V_n as a subset of $V_n' = V_n \times \mathbf{C}$ by sending v to $(v, 0)$ and hence we can consider $\phi_{b,n}$ as a map to $V_n' \times \mathbf{C}$.When $b = 0$, we let

$$\phi_{0,n}(s, z) = ((Q_r, s), z).$$

Then

$$\left(\frac{\partial \phi_{0,n}^*(g_{1,n}')}{\partial s}\right)(0, z) = 2.$$

q.e.d.

Theorem 4.6.2. *There is a $Q \in R_0[[\epsilon]]$ so that*

$$D_i(v^{(0)} - Q) \in \mathcal{I}_Q.$$

Proof. We have constructed a representation satisfying the hypotheses of Theorem 2.9.5. q.e.d.

Theorem 4.6.3. *The characteristic numbers of* $D_1, \dots, D_n$ *are* $1, 3, \dots, 2n-1$. *Further, the span of* $E_1^{[0]}, \dots, E_n^{[0]}$ *is the same of the span of* $\mathbf{K}_1, \dots, \mathbf{K}_n$, *where the* $E_k^{[0]}$ *are the leading terms of a normalized basis of the* $\mathbf{C}[[\epsilon]]$ *module* $\mathcal{M}$ *generated by the* D_k. (*Definition* 3.4.6.)

Proof. Using Lemma 2.8.3 and Lemma 4.6.1, we can construct a representation of $D_1, \dots, D_n$ satisfying the hypotheses of Corollary 3.4.7. q.e.d.

5. Poisson structures

5.1

Let

$$\hat{R}_0 = \mathbf{C}[\dots, \hat{a}_{-1}, \hat{a}_0, \hat{a}_1, \dots, \hat{b}_{-1}, \hat{b}_0, \hat{b}_1, \dots]$$

and let

$$\hat{R} = R_0[\zeta, \zeta^{-1}].$$

We say a monomial in the $\hat{a}_k$ and $\hat{b}_l$ has weight r if the sum of the subscripts of the $\hat{a}_k$ and $\hat{b}_l$ sum to r. So the monomial $\hat{a}_1\hat{a}_2\hat{b}_{-3}$ has weight 0, as does ζ. Let $I_k \subset R$ be the $\mathbf{C}$ span of all the elements of weight k. Let M_N be the ideal of R_0 generated by

$$\hat{a}_N, \hat{a}_{N+1}, \dots, \hat{a}_{-N}, \hat{a}_{-N-1}, \dots, \hat{b}_N, \hat{b}_{N+1}, \dots, \hat{b}_{-N}, \hat{b}_{-N-1},$$

i.e., a monomial is in M_N if it involves $\hat{a}_k$ or $\hat{b}_k$ with $|k| \geq N$. We also let M_N denote the induced ideal in $\hat{R}$. Let $\hat{I}_k$ be the completion of I_k with respect to subspaces $I_k \cap M_N$ as $N \to \infty$. Then

$$\mathcal{F} = \bigoplus_k \hat{I}_k$$

is called the Fourier ring. $\mathcal{F}$ is naturally a graded ring.

5.2

Suppose we are given elements f, g of $\mathcal{S} = \mathbf{C}[z, z^{-1}]$. We define $\hat{a}_n(f, g) \in \mathbf{C}$ to be the coefficient of z^n in f and $\hat{b}_n(f, g)$ to be the coefficient of z^n in g. If $P \in \mathcal{F}$, we can extend these definitions to define $P(f, g) \in \mathbf{C}[\zeta, \zeta^{-1}]$. So if $f = \sum \alpha_n z^n$ and $g = \sum \beta_n z^n$ and $P \in I_k$, then $P(f, g)$

is the result of substituting α_n for $\hat{a}_n$ and β_n for $\hat{b}_n$ in P. Note that $P(f,g)$ is well-defined. To check that two elements of $\mathcal{F}$ are equal, all we have to do is to check that they induce identical functions on $\mathcal{S}^2$. Also, if $P \in \mathcal{F}[Z]$, then we can define $P(f,g) \in \mathbf{C}[Z,\zeta,\zeta^{-1}]$. We denote by $\mathcal{F}_0$ the analogous construction for $\hat{R}_0$. We can naturally map $\Psi : \mathcal{F} \to \mathcal{F}_0[[\epsilon]]$ by sending ζ to $\exp(2\pi i\epsilon)$ considered as a formal power series in ϵ.

Suppose we have $f \in \mathcal{S}$ and let N be a positive integer, which we will think of as being large and let

$$\zeta_N = \exp\left(\frac{2\pi i}{N}\right).$$

Then we can define

$$T_N(f)(n) = f(\zeta_N^n)$$

so $T_N(f) : \mathbf{Z}/N \to \mathbf{C}$. Note that if N is sufficiently large depending on f and k, then we can recover f from $T_N(f)$, namely

$$\frac{1}{N}\sum T_N(f)(n)\zeta_N^{-nk}$$

is the coefficient of z^k.

Now suppose we have a polynomial $P \in S_1$ (see Definition 2.2.1 for definition of S_1) and let $\mathcal{C}_N$ be the set of $\mathbf{C}$ valued functions on $\mathbf{Z}$ periodic of order N. We can then define

$$P(F,G)(n) = P\Big(\dots, F(n-1), F(n), F(n+1), \dots; \dots, G(n-1), G(n), G(n+1)\Big) \tag{32}$$

Given $f \in \mathcal{C}_N$, we define

$$\hat{f}(n) = \frac{1}{N}\sum_{k\in\mathbf{Z}/N} f(k)\zeta_N^{-nk}.$$

Now suppose we are given two elements P_1, P_2 of S_1. We can find a continuous derivations $\mathcal{E}_{P_1,P_2}$ of $\mathcal{F}$ with the property that for all $f, g \in \mathcal{S}$ if $P_i(T_N(f), T_N(g)) = h_{i,N}$,

$$(\mathcal{E}_{P_1,P_2}(\hat{a}_n)(f,g))_{\zeta=\zeta_N} = \hat{h}_{1,N}(n)$$

and

$$\Big(\mathcal{E}_{P_1,P_2}(\hat{b}_n)(f,g)\Big)_{\zeta=\zeta_N} = \hat{h}_{2,N}(n)$$

for all N sufficiently large depending on f and g and n.

We can construct a series of maps

$$f_n : R[[\epsilon]] \to \mathcal{F}_1$$

with the properties

$$f_n(v^{(k)}) = (2\pi in)^k \hat{a}_n,$$
$$f_n(w^{(k)}) = (2\pi in)^k \hat{b}_n,$$
$$f_n(1) = \delta_{n,0}$$

and

$$f_n(FG) = \sum_{l \in \mathbf{Z}} f_l(F) f_{n-l}(G).$$

Then we have

$$f_n(\partial P) = 2\pi in f_n(P).$$

Suppose we are given a tame derivation D of $R[[\epsilon]]$ and a derivation $\hat{D}$ of $\hat{R}_0[[\epsilon]]$.

Definition 5.2.1. We say D and $\hat{D}$ are compatible if

$$f_n(D(P)) = \hat{D}(f_n(P)).$$

Proposition 5.2.2. *Given $D = \mathcal{D}_{P,Q}$ for $P, Q \in S_1$, then there is a unique compatible $\hat{D}$. $\hat{D}$ maps the image of Ψ to itself and restricts to $\mathcal{E}_{P,Q}$ on the image of Ψ.*

5.3

Suppose we have a finite collection $\mathcal{P}$ of elements of S_1

$$P_{-k}, \ldots, P_0, \ldots, P_k, Q_{-k}, \ldots, Q_0, \ldots, Q_k, R_{-k}, \ldots, R_0, \ldots, R_k.$$

Any polynomial with higher index is considered to be 0. Under some conditions on $\mathcal{P}$, we can attempt define a Poisson bracket $\{\ ,\ \}_{\mathcal{P}}$ on the functions $\mathcal{G}_N$ on $\mathcal{C}_N^2$ by asking that the bracket be a derivation in each slot, be anti-symmetric and satisfy Jacobi's identity. Further, let $A_k, B_k \in \mathcal{G}_N$ be defined by

$$A_k(f, g) = f(k)$$

and

$$B_k(f, g) = g(k).$$

Then we can define

$$\{A_k, A_l\}_{\mathcal{P}} = P_{k-l}(\ldots, \hat{A}_k, A_{k+1}, \ldots; \ldots, \hat{A}_l, \ldots)$$

$$\{A_k, B_l\}_{\mathcal{P}} = Q_{k-l}(\ldots, \hat{A}_k, A_{k+1}, \ldots; \ldots, \hat{B}_l, \ldots)$$

and

$$\{B_k, B_l\}_{\mathcal{P}} = R_{k-l}(\ldots, \hat{B}_k, B_{k+1}, \ldots; \ldots, \hat{B}_l, \ldots).$$

Note the ˆ in the above equations is the place holder. We will suppose that $\{\ ,\ \}_{\mathcal{P}}$ defines a Poisson bracket on $\mathcal{G}_N$ for N sufficiently large. We can define a modified bracket $\{\ ,\ \}_{\mathcal{P},N}$ by

$$\{A_k, B_l\}_{\mathcal{P},N}$$
$$= Q_{k-l}\left(\ldots, -2 + \frac{1}{N^2}A_k, -2 + \frac{1}{N^2}A_{k+1}, \ldots; \ldots, 1 + \frac{1}{N^2}B_l, \ldots\right).$$

Next define

$$\hat{A}_{k,N} = \frac{1}{N}\sum_{l\in\mathbf{Z}/N} \zeta_N^{-kl} A_l$$

and

$$\hat{B}_{k,N} = \frac{1}{N}\sum_{l\in\mathbf{Z}/N} \zeta_N^{-kl} B_l.$$

Proposition 5.3.1. *Suppose that* $\{\ ,\ \}_{\mathcal{P}}$ *defines a Poisson bracket on* $\mathcal{G}_N$ *for all* N *sufficiently large. Then there is a Poisson bracket* $\{\ ,\ \}_{\mathcal{P}} : \mathcal{F}\times\mathcal{F} \to \mathcal{F}[Z]$ *so that for* $f, g \in \mathcal{S}$, *then*

$$(\{\hat{a}_k, \hat{a}_l\}_{\mathcal{P}}(f,g))_{\zeta=\zeta_N, Z=1/N} = \{\hat{A}_{k,N}, \hat{A}_{l,N}\}_{\mathcal{P},N}(T_N(f), T_N(g))$$

for all N *sufficiently large with analogous formulas for* $\{\hat{a}_k, \hat{b}_l\}_{\mathcal{P}}(f,g)$ *and* $\{\hat{b}_k, \hat{b}_l\}_{\mathcal{P}}(f,g)$.

Given $P \in S_1[Z]$, define $P_N : \mathcal{C}_N \times \mathcal{C}_N \to \mathbf{C}$ by

$$P_N(F,G) = \frac{1}{N}\sum_{n\in\mathbf{Z}/N\mathbf{Z}} P\left(-2 + \frac{F}{N^2}, 1 + \frac{G}{N^2}\right)(n)_{Z=1/N}.$$

Lemma 5.3.2. *Suppose* $P \in S_1[Z]$. *Then there is a unique* H_P *in* $\mathcal{F}[Z]$ *so that*

$$H_P(f,g)_{\zeta=\zeta_N,\, Z=1/N} = P_N(T_N(f), T_N(g)).$$

for N *sufficiently large.*

Proposition 5.3.3.

$$(\{\hat{a}_k, H_P\}(f,g))_{\zeta=\zeta_N,\, Z=1/N} = \{\hat{A}_k, P_N\}(T_N(f), T_N(g))$$

and

$$(\{\hat{b}_k, H_P\}(f,g))_{\zeta=\zeta_N,\, Z=1/N} = \{\hat{B}_k, P_N\}(T_N(f), T_N(g)).$$

Suppose that $P \in S_1[Z]$ and f and g are in $\mathcal{S}$. The function of ϵ defined by

$$H(\epsilon) = P(f,g)_{\zeta=\exp(2\pi i\epsilon),\, Z=\epsilon}$$

is an analytic function of ϵ and for N sufficiently large,

$$H\left(\frac{1}{N}\right) = P_N(T_N(f), T_N(g)).$$

If

$$\left|H\left(\frac{1}{N}\right)\right| < K(f,g)N^{-l}, \tag{33}$$

for all N sufficiently large, then the first $l-1$ derivatives of H vanish. We can attach a formal power series $\Psi(P)$ to P in $\mathcal{F}_0[[\epsilon]]$ by setting $Z=\epsilon$ and setting $\zeta = 1 + 2\pi i\epsilon + \cdots = \exp(2\pi i\epsilon)$. If (33) holds for all $f, g \in \mathcal{S}$, then the first $l-1$ derivatives of $\Psi(L)$ vanish.

5.4

We next calculate two Poisson brackets. Our first bracket is given by:

$$\{B_n, A_n\}_{\mathcal{P}_1} = -2B_n$$
$$\{B_{n-1}, A_n\}_{\mathcal{P}_1} = 2B_{n-1},$$

with all other brackets between the A_i and B_j being zero, except for the obvious antisymmetric versions of these two formulas. We can now compute:

$$\begin{aligned}
\{\hat{A}_{n,N}, \hat{B}_{m,N}\}_{\mathcal{P}_1} &= \frac{1}{N^2}\left\{\sum_l \zeta_N^{-ln} A_l, \sum_k \zeta_N^{-mk} B(k)\right\} \\
&= \frac{1}{N^2}\sum_{k,l} \zeta_N^{-(ln+mk)}\{A_l, B_k\} \\
&= \frac{1}{N^2}\sum_l \zeta_N^{-l(n+m)} 2B_l - \sum_k \zeta_N^{-k(n+m)+n} 2B_k \\
&= \frac{2}{N}\hat{B}_{n+m}(1-\zeta_N^n),
\end{aligned} \tag{34}$$

and all the other brackets zero except as obviously required by antisymmetry.

For our second Poisson bracket $\{\ ,\ \}_{\mathcal{P}_2}$ we take

$$\{A_k, A_{k+1}\}_{\mathcal{P}_2} = B_k$$

$$\{B_k, A_{k+1}\}_{\mathcal{P}_2} = B_k A_{k+1}$$

$$\{B_k, A_k\}_{\mathcal{P}_2} = -B_k A_k$$

$$\{B_k, B_{k+1}\}_{\mathcal{P}_2} = B_k B_{k+1}.$$

Now we can calculate the appropriate Fourier brackets:

(35)

$$\begin{aligned}
&\{\hat{B}_{n,N}, \hat{B}_{m,N}\} \\
&= \frac{1}{N^2}\left\{\sum_l B_l \zeta^{-lk}, \sum_k B_k \zeta^{-km}\right\} \\
&= \frac{1}{N^2}\sum_{k,l}\{B_l, B_k\}\zeta_N^{-ln-km} \\
&= \frac{1}{N^2}\sum_k \{B_{k+1}, B(k)\}\zeta_N^{-(k+1)n-km} + \sum_k \{B_{k-1}, B_k\}\zeta_N^{-(k-1)n-km} \\
&= \frac{1}{N^2}\sum_k B_k B_{k-1}\zeta^{-(k-1)n-km} - B_k B_{k+1}\zeta^{-(k+1)n-km} \\
&= \frac{1}{N^2}\sum_k B_k B_{k+1}(\zeta^{-kn-(k+1)m} - \zeta^{-(k+1)n-km}) \\
&= \frac{1}{N^2}\sum_{k,r,s} \hat{B}_r \hat{B}_s(\zeta^{(-kn-(k+1)m)+rk+s(k+1)} - \zeta^{-(k+1)n-km+rk+s(k+1)}) \\
&= \frac{1}{N}\sum_{n+m=r+s} \hat{B}_r \hat{B}_s(\zeta^{-m+s} - \zeta^{-n+s}).
\end{aligned}$$

Next,

(36)

$$\begin{aligned}
&\{\hat{B}_{n,N}, \hat{A}_{m,N}\} \\
&= \frac{1}{N^2}\left\{\sum_l B_l \zeta^{-lk}, \sum_k A_k \zeta^{-km}\right\} \\
&= \frac{1}{N^2}\sum_{k,l}\{B_l, A_k\}\zeta_N^{-ln-km}
\end{aligned}$$

$$= \frac{1}{N^2}\left(\sum_l \{B_l, A_{l+1}\}\zeta_N^{-(ln+(l+1)m)} + \sum_l \{B_l, A_l\}\zeta_N^{-ln-lm}\right)$$
$$= \frac{1}{N^2}\left(\sum_l B_l A_{l+1}\zeta_N^{-(ln+(l+1)m)} - \sum_l B_l A_l\zeta_N^{-ln-lm}\right)$$
$$= \frac{1}{N^2}\left(\sum_l B_l(A_{l+1}\zeta_N^{-m} - A_l)\zeta_N^{-ln-lm}\right)$$
$$= \frac{1}{N^2}\sum_{l,r,s} \hat{B}_r\hat{A}_s\left(\zeta_N^{-m+rl+s(l+1)-ln-lm} - \zeta_N^{+rl+sl-ln-lm}\right)$$
$$= \frac{1}{N}\sum_{r+s=n+m} \hat{B}_r\hat{A}_s\left(\zeta_N^{-m+s} - 1\right).$$

Finally, we compute

$$(37)\qquad \{\hat{A}_{n,N}, \hat{A}_{m,N}\}$$
$$= \frac{1}{N^2}\left\{\sum_l A_l\zeta_N^{-lk}, \sum_k A_k\zeta_N^{-km}\right\}$$
$$= \frac{1}{N^2}\sum_{k,l}\{A_l, A_k\}\zeta_N^{-ln-km}$$
$$= \frac{1}{N^2}\left(\sum_l \{A_l, A_{l+1}\}\zeta_N^{-ln-(l+1)m} + \{A_l, A_{l-1}\}\zeta_N^{-ln-(l-1)m}\right)$$
$$= \frac{1}{N^2}\left(\sum_l B_l\zeta_N^{-ln-(l+1)m} - B_{l-1}\zeta_N^{-ln-(l-1)m}\right)$$
$$= \frac{1}{N^2}\left(\sum_{l,r} \hat{B}_r\zeta_N^{-ln-(l+1)m+rl} - \hat{B}_r\zeta_N^{-ln-(l-1)m+r(l-1)}\right)$$
$$= \frac{1}{N}\hat{B}_{n+m}(\zeta_N^{-m} - \zeta_N^{-n}).$$

We will now investigate the bracket $\{\ ,\ \}_2$ defined by $\{\ ,\ \}_2 = \{\ ,\ \}_{\mathcal{P}_2}$. Note that if we have a continuous bracket

$$\{\ ,\ \} : \mathcal{F}\times\mathcal{F} \to \mathcal{F}[Z]$$

then we have an induced bracket on $\mathcal{F}_0[[\epsilon]]$ obtained by replacing Z by ϵ and ζ by the formal power series $\exp(2\pi i\epsilon)$. By abuse of notation, we continue to call the induced bracket by the some name as the original bracket.

Let $W_l = Y_0^l$ and consider

$$X_R = \sum_{l=1}^{R} \frac{(-1)^l H_{W_l} Z^{2l-2}}{l+1}.$$

Proposition 5.4.1. *Let*

$$X = \lim_{R\to\infty} \Psi(X_R).$$

Then X is a Casimir for $\{\ ,\ \}_{\mathcal{P}_2}$. The leading term of X in ϵ is $\hat{b}_0$.

Proof. We will look at $\{\hat{a}_p, X_R\} = V_R$. Now

$$V_R(f,g)_{\zeta=\zeta_N,\, Z=1/N} = \sum_{r\in\mathbf{Z}/N\mathbf{Z}} \left(\sum_{l=0}^{R-1} \left(\frac{(-1)^l g(\zeta_N^r)^l}{N^{2l}} \right) \{\hat{A}_p, B_r\}_{\mathcal{P}_2,\,N}(T_N(f), T_N(g)) \right).$$

On the other hand,

$$\prod_{k=0}^{N} \left(1 + \frac{B_k}{N^2}\right)$$

is a Casimir for $\{\ ,\ \}_{\mathcal{P}_2,N}$, as was pointed out to me by Ali Kisisel. In particular for N sufficiently large,

$$\begin{aligned} (38)\qquad 0 &= \sum_{r=0}^{N} \frac{\{\hat{A}_p, B_r\}_{\mathcal{P}_2,N}}{1+\frac{B_r}{N^2}} (T_N(f), T_N(g)) \\ &= \sum_{r=0}^{N} \frac{\{\hat{A}_p, B_r\}_{\mathcal{P}_2,N}}{1+\frac{g(\zeta_N^r)}{N^2}} (T_N(f), T_N(g)) \\ &= \sum_{r=0}^{N} \left(\sum_{l=0}^{\infty} \left(\frac{(-1)^l g(\zeta_N^r)^l}{N^{2l}} \right) \{\hat{A}_p, B_r\}_{\mathcal{P}_2,N}(T_N(f), T_N(g)) \right). \end{aligned}$$

Thus if we fix f and g and l, we can find a constant $K(f,g)$ so that

$$|V_R(f,g)_{\zeta=\zeta_N,\, Z=1/N}| < K(f,g) N^{-l}$$

for N sufficiently large. Thus the first $l-1$ derivatives of $V_R(f,g)$ vanish. Since this is true of any f and g, the first $l-1$ derivatives of V_R vanish.

Thus $\{\hat{a}_p, X\} = 0$. A similar argument shows that $\{\hat{b}_p, X\} = 0$, so X is a Casimir. q.e.d.

Suppose that P_1, P_2 and P are in $S_1[Z]$. Suppose further that

$$\{A_k, P_N\}_{\mathcal{P},N} = P_1(\dots, A_{k-1}, \hat{A}_k, A_{k+1}, \dots; \dots, B_{k-1}, \hat{B}_k, B_{k+1}, \dots)$$

and

$$\{B_k, P_N\}_{\mathcal{P},N} = P_2(\dots, A_{k-1}, \hat{A}_k, A_{k+1}, \dots; \dots, B_{k-1}, \hat{B}_k, B_{k+1}, \dots),$$

with the $\hat{}$ indicating place holder, not Fourier.

Proposition 5.4.2. $\mathcal{E}_{P_1,P_2}(\hat{a}_p) = \{\hat{a}_p, H_P\}_{\mathcal{P}}$ *with a similar formula for* $\hat{b}_k$.

5.5

We get a series of derivations $\mathbf{D}_k$ of $\mathcal{F}[[\epsilon]]$ compatible with D_k (see Proposition 2.4.1). The $\mathbf{D}_k$ all preserve the ideal I_Q generated by

$$f_n(v^{(0)} - Q) = L_n.$$

On the other hand, we have two compatible Poisson brackets on $\mathcal{F}_0[[\epsilon]]$. Let

$$Z(n) = \exp(2\pi i n\epsilon).$$

The first is defined by

$$\{\hat{a}_n, \hat{b}_m\}_1 = (\delta_{n,-m} + \epsilon^2 \hat{b}_{n+m})(1 - \exp(2\pi\epsilon i n))$$

with all other terms zero except as dictated by the Poisson bracket axioms. Thus we obtain

$$\{\hat{a}_n, \hat{b}_m\}_1 = (-2\pi i\epsilon n)\delta_{n,-m} + \text{higher order terms in } \epsilon.$$

In particular,

$$\{L_n, L_{-n}\}_1 = \delta_{n,-m}(-4\pi\epsilon i n) + \text{higher order terms in } \epsilon.$$

The second is defined by

(39)

$$\begin{aligned}
&\{\hat{b}_n, \hat{b}_m\}_2 \\
&= \frac{1}{\epsilon} \sum_{n+m=r+s} \Phi'(\hat{b}_r)\Phi'(\hat{b}_s)(\exp(2\pi\epsilon i(-m+s) - \exp(2\pi\epsilon i(-n+s)) \\
&= \frac{1}{\epsilon^2}\{\hat{b}_n, \hat{b}_m\}_{\mathcal{P}_2}
\end{aligned}$$

with analogous expression for $\{\hat{b}_n, \hat{a}_m\}_2$ and $\{\hat{a}_n, \hat{a}_m\}_2$ from the Fourier expressions for the second bracket $\{\ ,\ \}_{\mathcal{P}_2}$. First suppose that $n+m=0$. We can then compute

(40)
$$\epsilon\{\hat{b}_n, \hat{b}_{-n}\}_2 = (\exp(2\pi\epsilon i(n) - \exp(2\pi\epsilon i(-n))(1+\epsilon^2\hat{b}_0)^2 + \sum_{r+s=0, r\neq 0} \epsilon^4 \hat{b}_r \hat{b}_s (\exp(2\pi\epsilon i(n+s) - \exp(2\pi\epsilon i(-n+s)).$$

If $n+m \neq 0$, then

$$\epsilon\{\hat{b}_n, \hat{b}_m\}_2 = \epsilon^2 \hat{b}_{n+m}(Z(n) - Z(-n) - Z(m) + Z(-m)) + \text{higher order terms in } \epsilon. \tag{41}$$

So we get

$$\{\hat{b}_n, \hat{b}_m\}_2 = \delta_{n+m,0}(4\pi i n) + \text{higher order terms in } \epsilon.$$

We have similar results of $\{\hat{a}_n, \hat{b}_m\}_2$ and $\{\hat{a}_n, \hat{a}_m\}_2$.

Now

$$L_n = \hat{a}_n - \hat{b}_n + \text{higher order terms in } \epsilon.$$

So

$$\{L_n, L_{-n}\}_2 = 16\pi i n \delta_{n,-m} + \text{higher order terms in } \epsilon.$$

5.6

Ideally, our object would be to define induced brackets on $\mathcal{F}[[\epsilon]]/I_Q = \mathcal{F}_0[[\epsilon]]$. We will define brackets on a somewhat different ring $\mathcal{S}$. First, let J_Q be the ideal generated by the L_q for $q \neq 0$ and I'_Q the ideal generated by J_Q and the Casimir

$$X_1 = \frac{X}{\epsilon^2}.$$

Our aim is to define a well-defined bracket $\{\ ,\ \}_2$ on $\mathcal{F}_0[[\epsilon]]/I'_Q$. Now for each $t \in \mathbf{C}$, the ring $\mathcal{F}[[\epsilon]]$ has an automorphism ϕ_t defined by

$$\phi(\hat{a}_n) = \hat{a}_n + t\delta_{n,0}$$

together with

$$\phi(\hat{b}_n) = \hat{b}_n.$$

Our first bracket behaves in the following way under ϕ:

$$\{\phi(a), \phi(b)\}_1 = \phi\{a, b\}_1.$$

Further, there are $H_k \in \mathcal{F}[[\epsilon]]$ so that the vector fields obtained by bracketing with the H_k with either bracket have the same span as the derivations of $\mathcal{F}[[\epsilon]]$ compatible with $\mathcal{D}_{T_k}$ (see Equation (3)). Further, the H_k all commute with respect to either bracket and because of Theorem 1.0.1, bracketing with H_k with respect to either bracket preserves the ideal I_Q. Further,

$$\phi_t(H_k) = H_k + \sum_{l<k} C_{k,l}(t) H_l,$$

where the $C_{k,l} \in \mathbf{C}[[\epsilon]][t]$.

Proposition 5.6.1. *Bracketing with the H_k with respect to either bracket preserves the ideal J_Q.*

Proof. Note that $\phi_t(L_n) = L_n + t\delta_{n,0}$. We want to show that $\{H_k, L_p\}_1 \in J_Q$ for $p \neq 0$. We work by induction on k. We have that the first H, namely $H_1 = \hat{a}_0$ is a Casimir for the first bracket. So we have the first step of the induction. But modulo J_Q, $G = \{H_k, L_p\}_1$ is then invariant under ϕ_t. But $G = XL_0 + Y$ where $Y \in J_Q$. Let $\mathcal{J}_N$ be the closure of the ideal generated by J_Q and all the $\hat{b}_m$ for $|m| > N$. $\phi_t(G)$ reduces to a polynomial of positive degree in t in the ring $\mathcal{F}[[\epsilon]]/\mathcal{J}_N[t]$ if $X \notin \mathcal{J}_N$. Note $\mathcal{F}[[\epsilon]]/\mathcal{J}_N$ is an integral domain. In fact, $\mathcal{F}[[\epsilon]]/\mathcal{J}_N$ is just $\mathbf{C}[\hat{a}_0, \hat{b}_{-N}, \ldots, \hat{b}_N][[\epsilon]]$. But $\phi_t(G)$ is invariant, so $X \in \mathcal{J}_N$ for all N, which implies that $X \in J_Q$ and so $G \in J_Q$. q.e.d.

If $P \in \mathcal{F}_0[[\epsilon]]$ and $\overline{P} \in \mathcal{F}_0[[\epsilon]]/J_Q$ is the image of P, then we define a good extension of $\overline{P}$ modulo ϵ^n to be an element $P' \in \mathcal{F}_0[[\epsilon]]$ so that $\{L_q, P'\}_k \in (J_Q + \epsilon^n)$ for $q \neq 0$. An extension is good if it is good modulo ϵ^n for all positive n. It is easy to see that good extensions exist. Suppose we have constructed a good extension P_n modulo ϵ^n. Then we can try

$$P_{n+1} = P_n + \epsilon^n \sum_{q \neq 0} W_q L_q$$

for $W_q \in \mathcal{F}_0$. Bracketing through by L_r for $r \neq 0$ allows us to choose the W_q uniquely so that P_{n+1} is good modulo ϵ^{n+1}. Thus we can find a good extension of $\overline{P}$. Given $\overline{P}, \overline{P}' \in \mathcal{F}_0[[\epsilon]]/J_Q$, we can define their bracket by taking good extensions P and P' of $\overline{P}$ and $\overline{P}'$ and taking their bracket in $\mathcal{F}[[\epsilon]]$ and then reducing modulo J_Q. This construction gives a well-defined bracket $\{P, Q\}_2$ on $\mathcal{F}[[\epsilon]]/J_Q$. Now I'_Q is obtained from J_Q by adding the Casimir X_1 so we get an induced bracket on $\mathcal{F}[[\epsilon]]/I'_Q$. For a given $n \neq 0$, define

$$\hat{\beta}_n = \hat{b}_n + \left(\frac{1}{2} + \frac{n\epsilon i\pi}{2} - \frac{\epsilon^2\pi^2 n^2}{4}\right) L_n + \epsilon^2 \sum_{k\neq 0} \left(-\frac{n}{8} + \frac{3k}{8}\right) \hat{b}_{n-k} \frac{L_k}{k}$$

is the good extension of $\hat{b}^n$ modulo ϵ^3 for the second bracket. For $n = 0$, let

$$\hat{\beta}_0 = \frac{1}{2}\left(\hat{a}_0 + \hat{b}_0\right) + \frac{1}{2}\sum_{k\neq 0} L_k \hat{b}_{-k}.$$

Theorem 5.6.2.

$$\{\hat{\beta}_n, \hat{\beta}_m\}_2 \equiv i\left(\pi(n-m)\hat{\beta}_{n+m} - \delta_{n,-m}\pi^3 n^3 \mod \epsilon^3\right).$$

The ring $\mathcal{F}[[\epsilon]]/I'_Q$ is generated topologically by the images of $\hat{\beta}_k$ and ϵ. Further, all the H_k still Poisson commute in this extension of the Virasoro algebra.

6. Convergence?

6.1

Let Q be the element of $R_0[[\epsilon]]$ we have constructed in Theorem 4.6.2. As in [5], we can compute the coefficients $Q_n \in R_0$ of Q. If g is a periodic function analytic on $\mathbf{R}$, we can ask when the power series

$$\sum_{k=0}^{\infty} Q_n(g)(z)\epsilon^n \tag{42}$$

converges for $z \in \mathbf{R}$. Suppose that

$$|g^{(n)}(z)| < n!, \tag{43}$$

where $g^{(n)}$ indicates the n^{th} derivative of g. For $n \leq 23$, I calculated $Q_n(g)$ using Maple and got a bound $|Q_n(g)(z)| < K_n$ for $z \in \mathbf{R}$ by replacing each of the terms in Q_n by the obvious estimate using (43). Here are the decimal values of K_n.

n	K_n
2	.500
3	.375
4	.359
5	.312
6	.300
7	.289
8	.283
9	.288
10	.285
11	.305
12	.312
13	.348
14	.387
15	.452
16	.634
17	.756
18	1.70
19	1.95
20	7.81
21	8.46
22	53.2
23	55.2

In order for 42 to converge for $\epsilon < 1/T$ all we would need is that $K_n < T^n$. On the basis of the fact that we have constructed many functions g coming from algebraic geometry for which 6.1.1 converges and the fact that the K_n appear to be growing not too fast, I believe there should be some general convergence property of Q. (Calculating the case $n = 23$ used over a gigabyte of memory and took over 500 hours on a Sun Enterprise.)

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Department of Mathematics
UCLA